

# Heavy Tails and Financial Time Series Models

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# Outline

## Financial time series modeling

- General comments
- Characteristics of financial time series
- Examples (exchange rate, Amazon)
- Multiplicative models for log-returns (GARCH, SV)

## Regular variation

- univariate case
- multivariate case

## Applications of regular variation

- Stochastic recurrence equations (GARCH)
- Stochastic volatility
- Time-reversibility
- Point process convergence
- Extremes and extremal index
- Limit behavior of sample correlations

## Wrap-up

# Financial Time Series Modeling

**One possible goal:** Develop models that capture essential features of financial data.

**Strategy:** Formulate families of models that at least exhibit these key characteristics. (e.g., GARCH and SV)

**Linkage with goal:** Do fitted models actually capture the desired characteristics of the real data?

**Answer wrt to GARCH and SV models:** Yes and no. Answer may depend on the features.

**Stărică's paper:** “Is GARCH(1,1) Model as Good a Model as the Nobel Accolades Would Imply?”

This paper discusses inadequacy of GARCH(1,1) model as a “data generating process” for the data.

## Financial Time Series Modeling (cont)

**Goal of this talk:** compare and contrast some of the features of GARCH and SV models.

- Regular-variation of finite dimensional distributions
- Time-reversibility
- Point process convergence
- Extreme value behavior
- Sample ACF

## Characteristics of financial time series

Define  $X_t = \ln(P_t) - \ln(P_{t-1})$  (log returns)

- heavy tailed

$$P(|X_1| > x) \sim RV(-\alpha), \quad 0 < \alpha < 4.$$

- uncorrelated

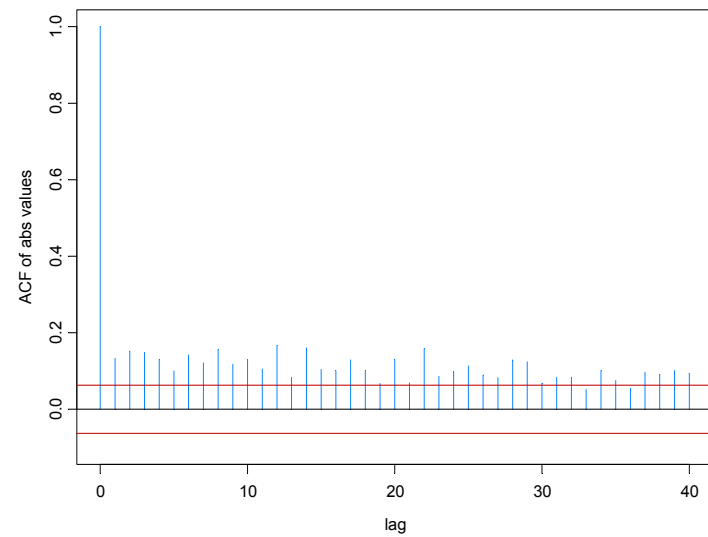
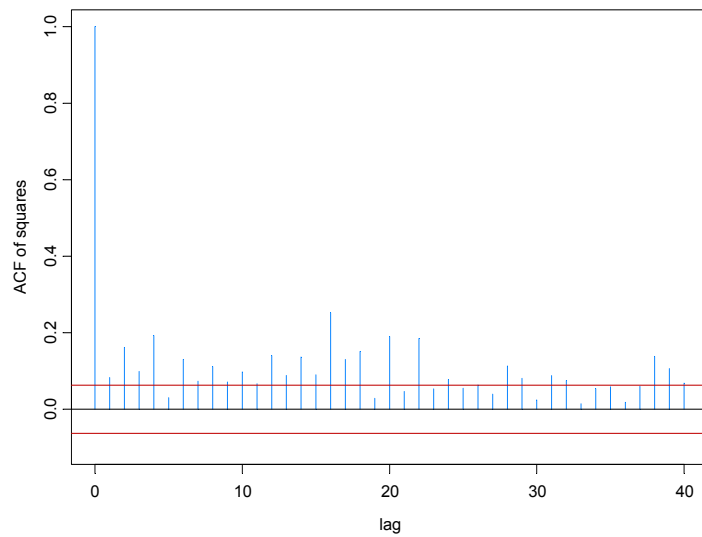
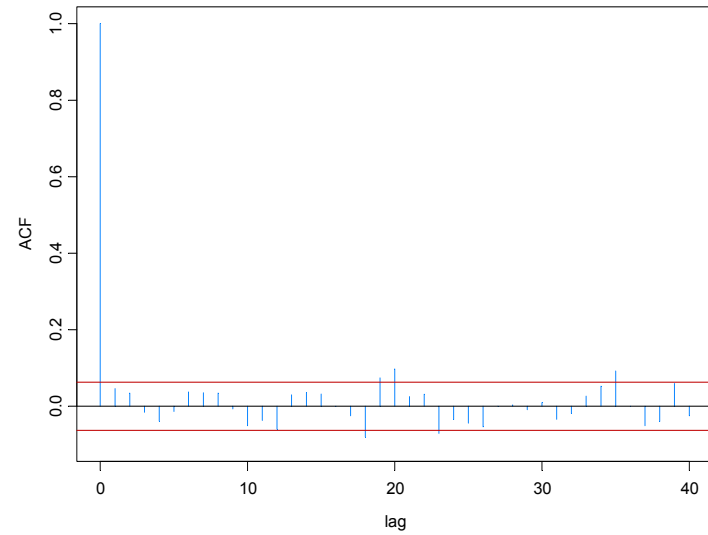
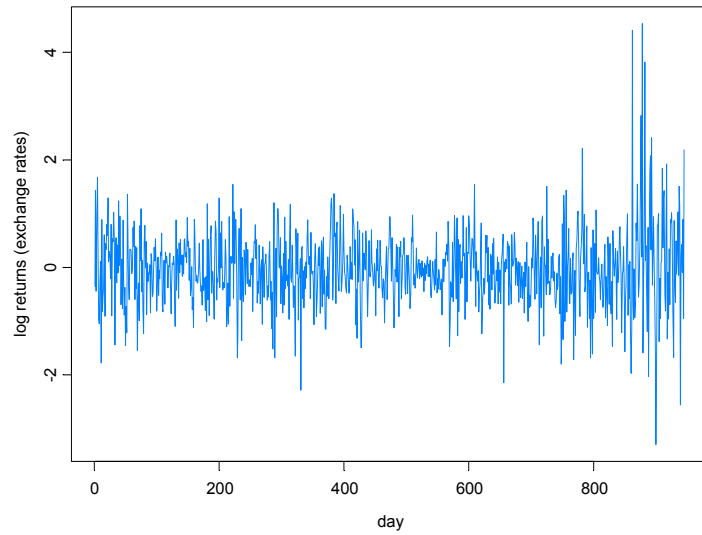
$$\hat{\rho}_X(h) \text{ near } 0 \text{ for all lags } h > 0$$

- $|X_t|$  and  $X_t^2$  have slowly decaying autocorrelations

$$\hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to } 0 \text{ slowly as } h \text{ increases.}$$

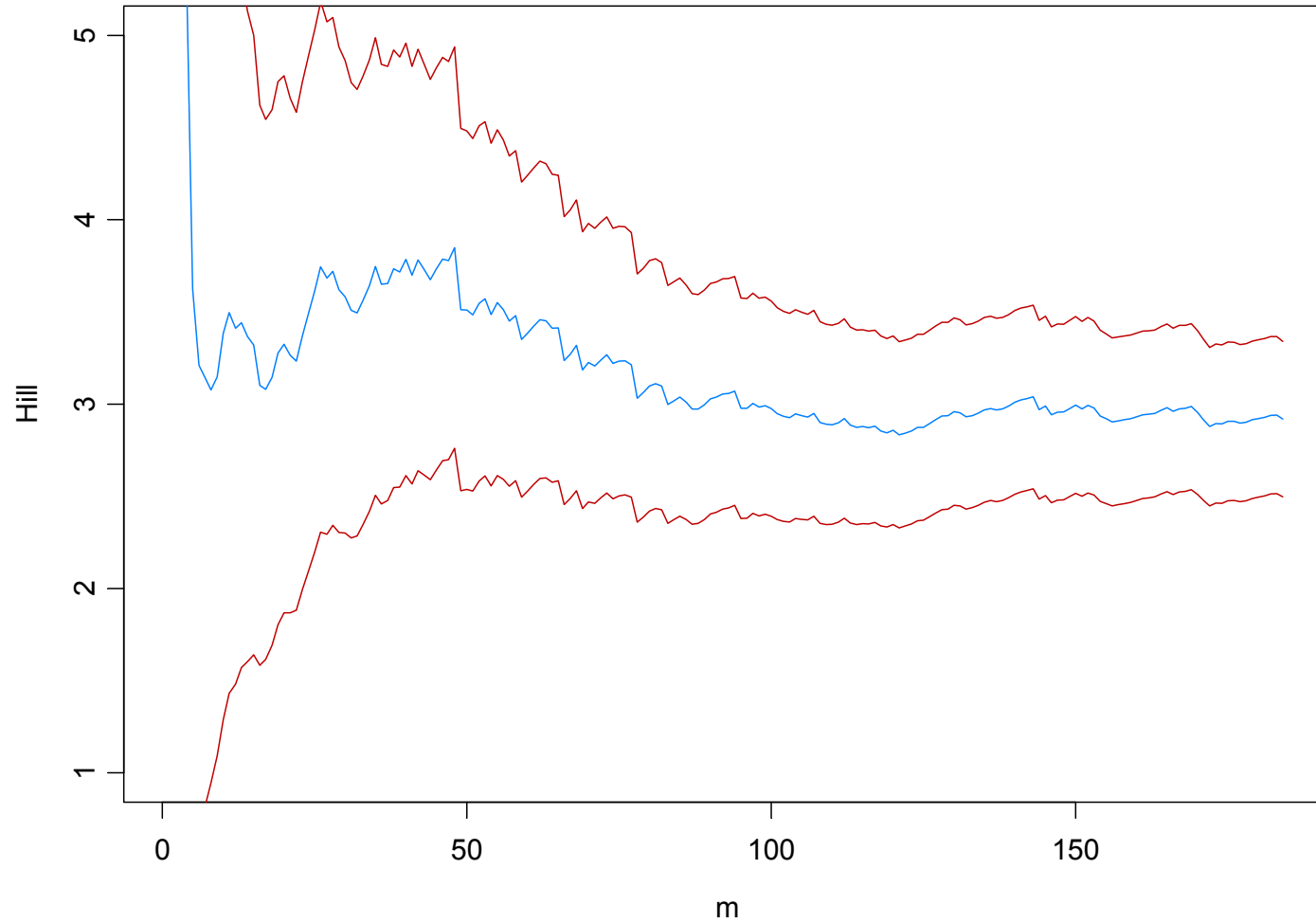
- process exhibits 'volatility clustering'.

## Example: Pound-Dollar Exchange Rates (Oct 1, 1981 – Jun 28, 1985; Koopman website)



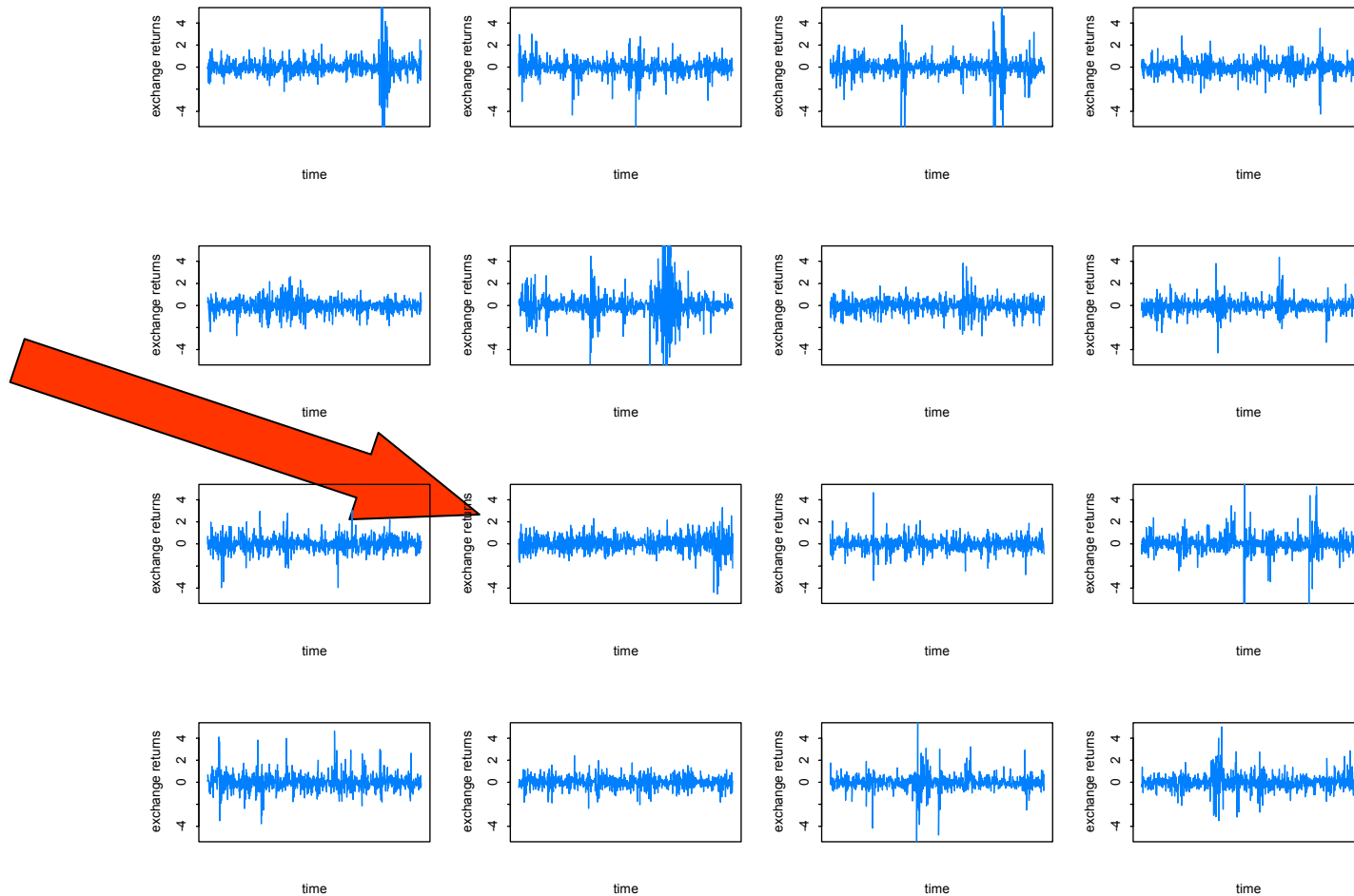
## Example: Pound-Dollar Exchange Rates

### Hill's estimate of alpha (Hill Horror plots-Resnick)



## Stărică Plots for Pound-Dollar Exchange Rates

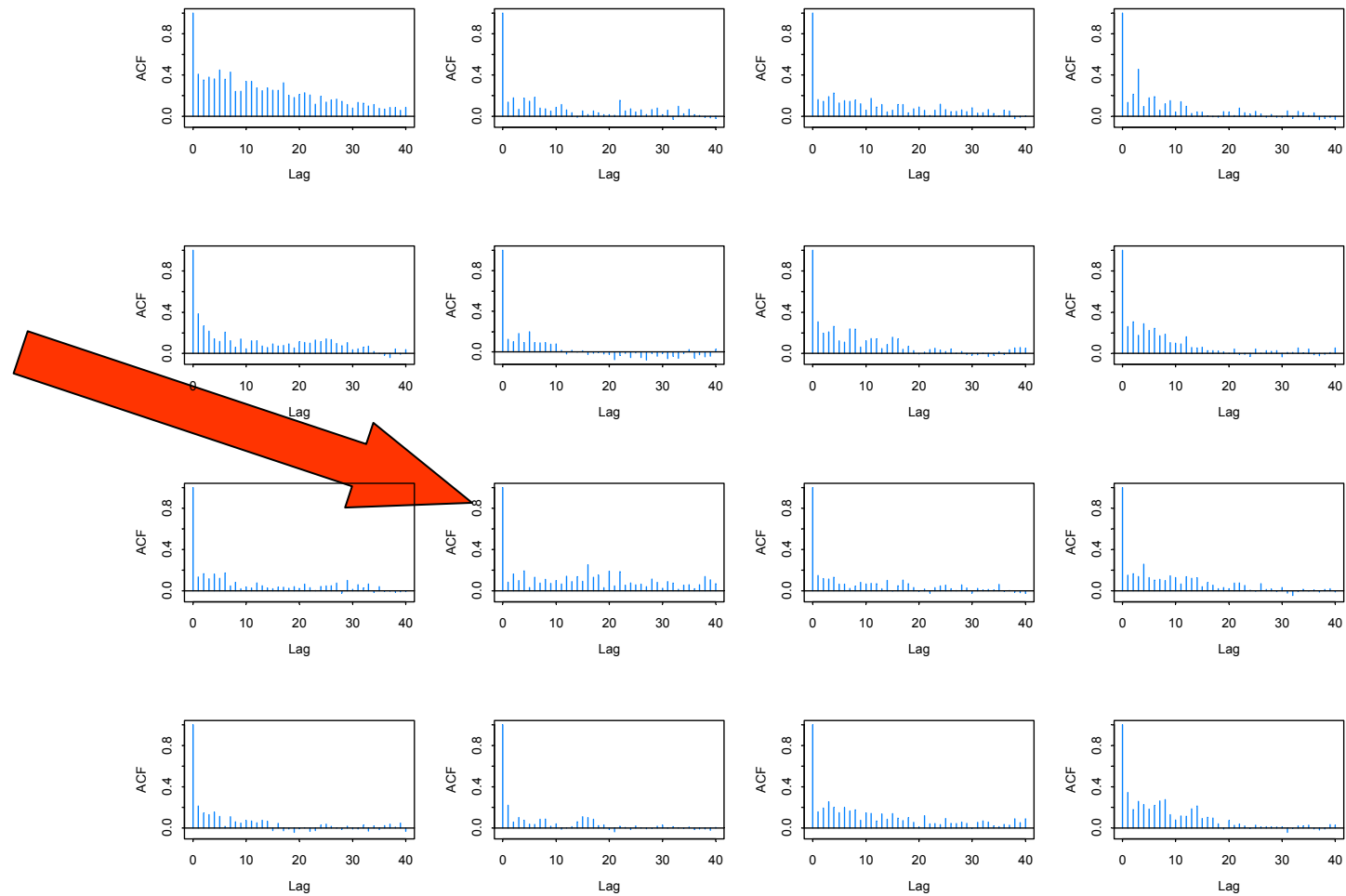
15 realizations from GARCH model fitted to exchange rates + exchange rate data. Which one is the real data?





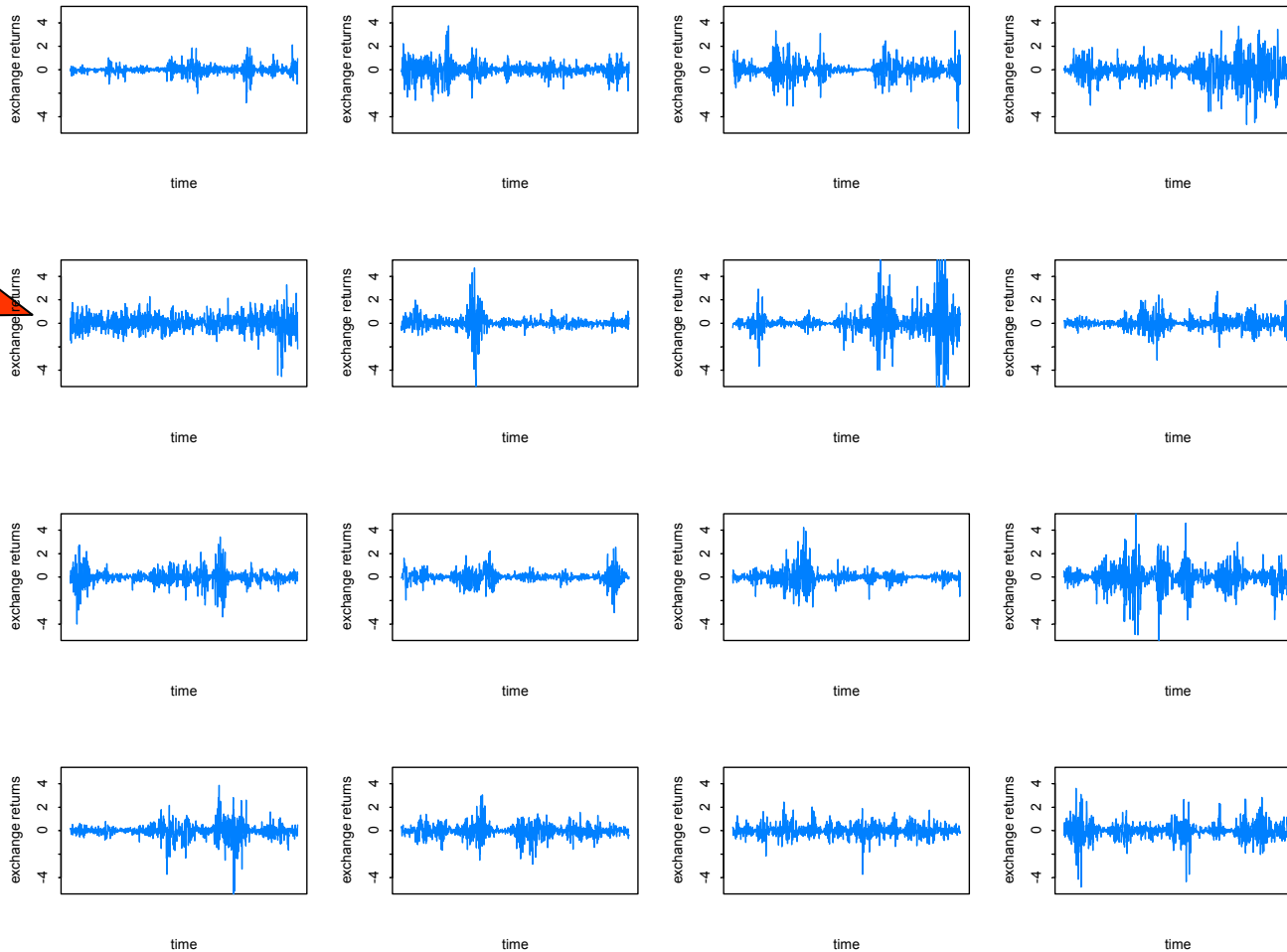
## Stărică Plots for Pound-Dollar Exchange Rates

ACF of the squares from the 15 realizations from the GARCH model on previous slide.

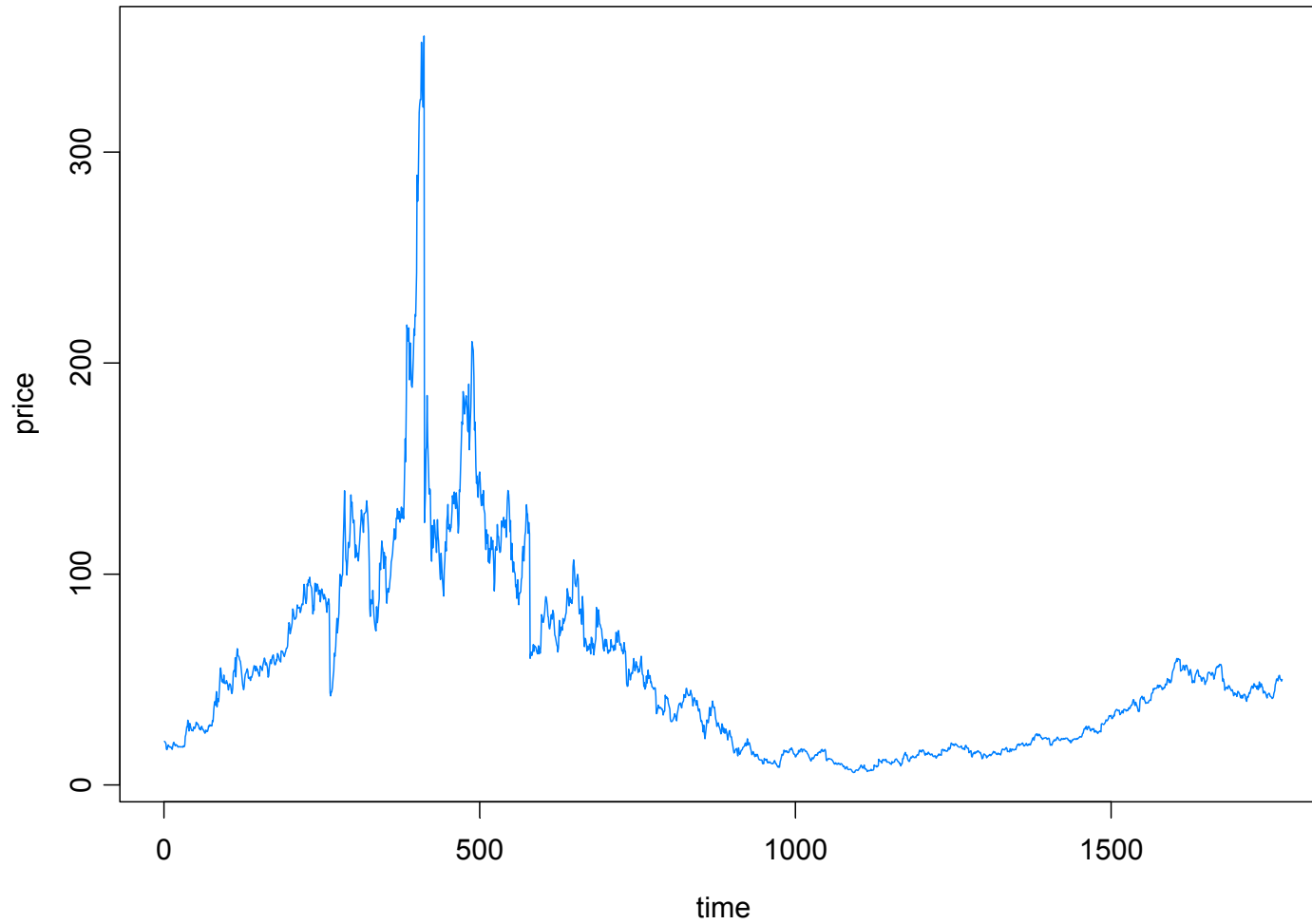


## Stărică Plots for Pound-Dollar Exchange Rates

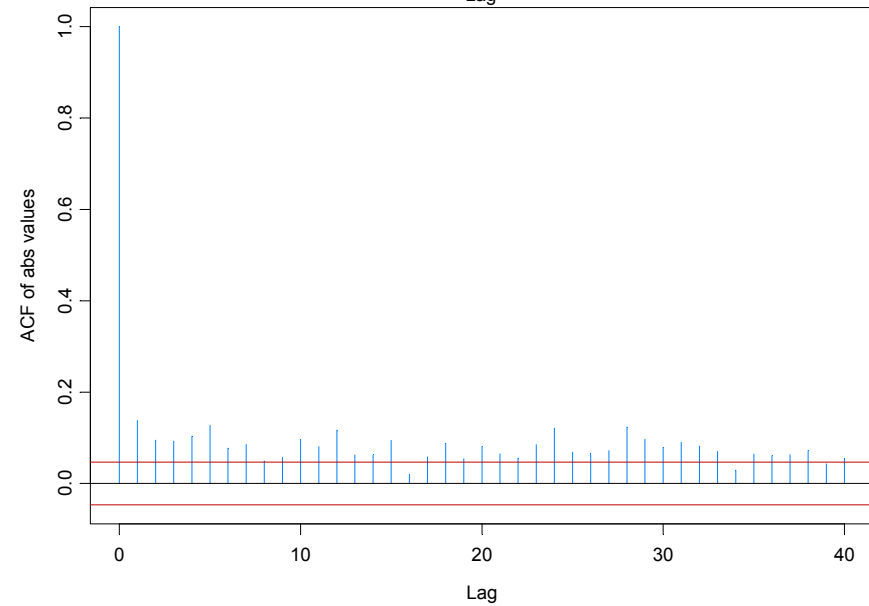
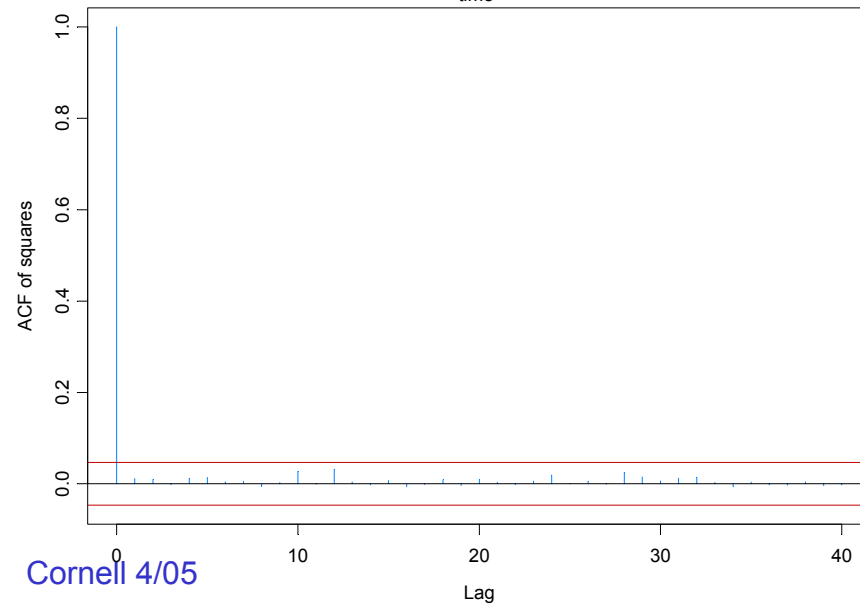
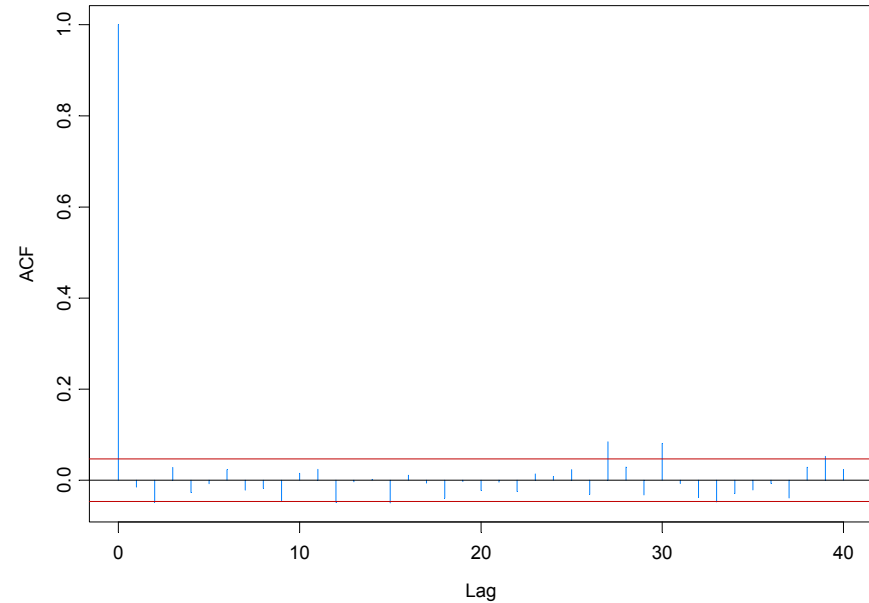
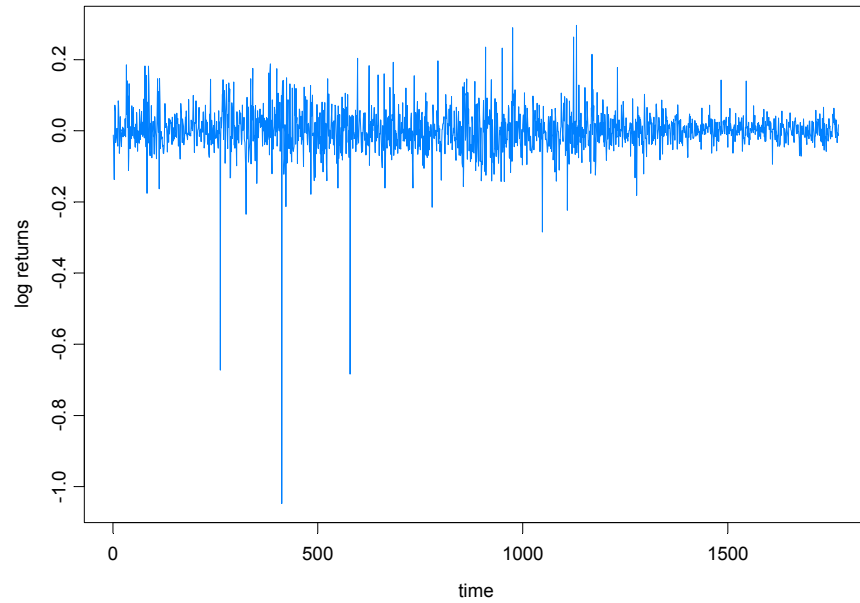
15 realizations from SV model fitted to exchange rates + real data. Which one is the real data?



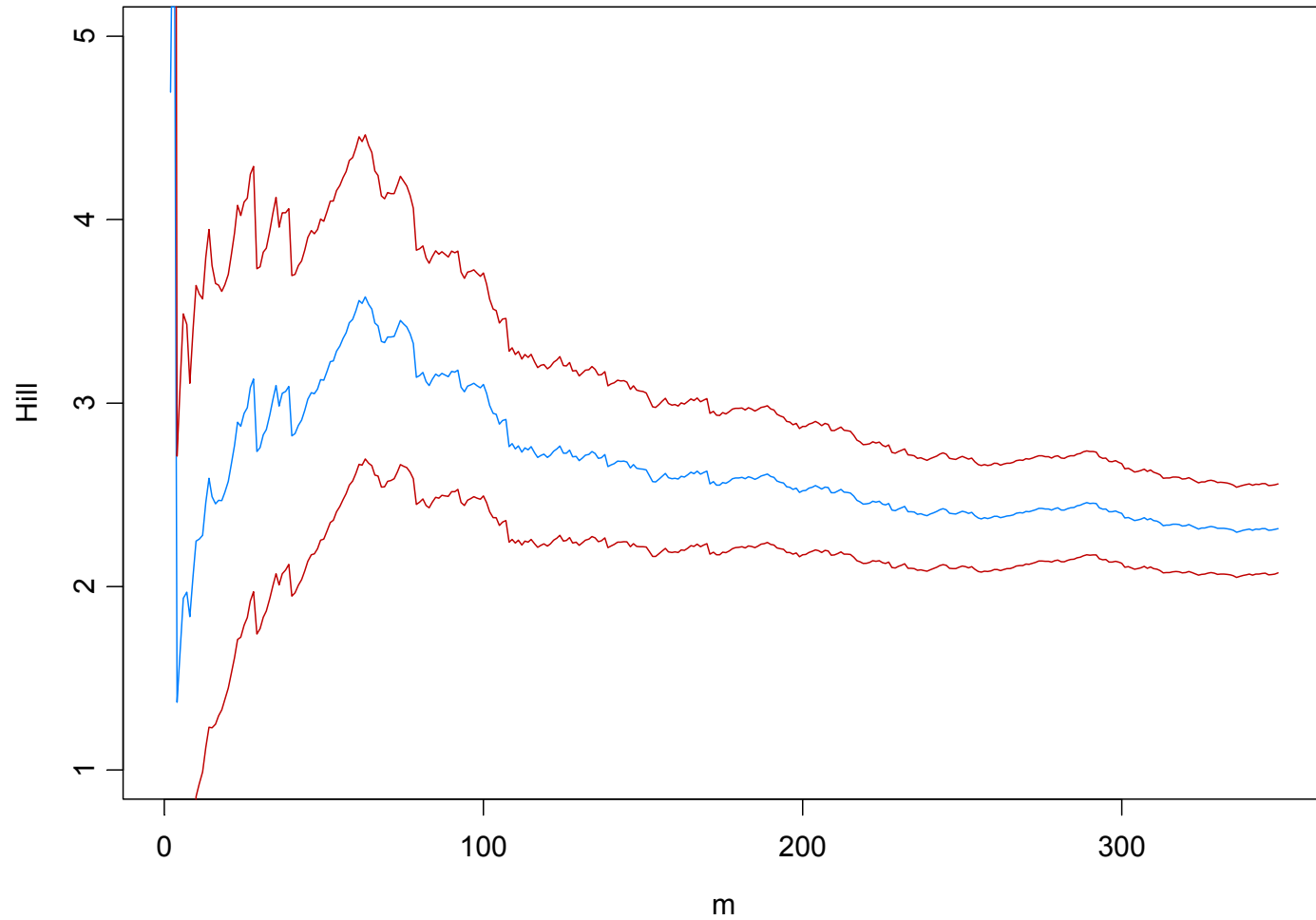
Example: Amazon (May 16, 1997 – June 16, 2004)



# Example: Amazon-returns (May 16, 1997 – June 16, 2004)

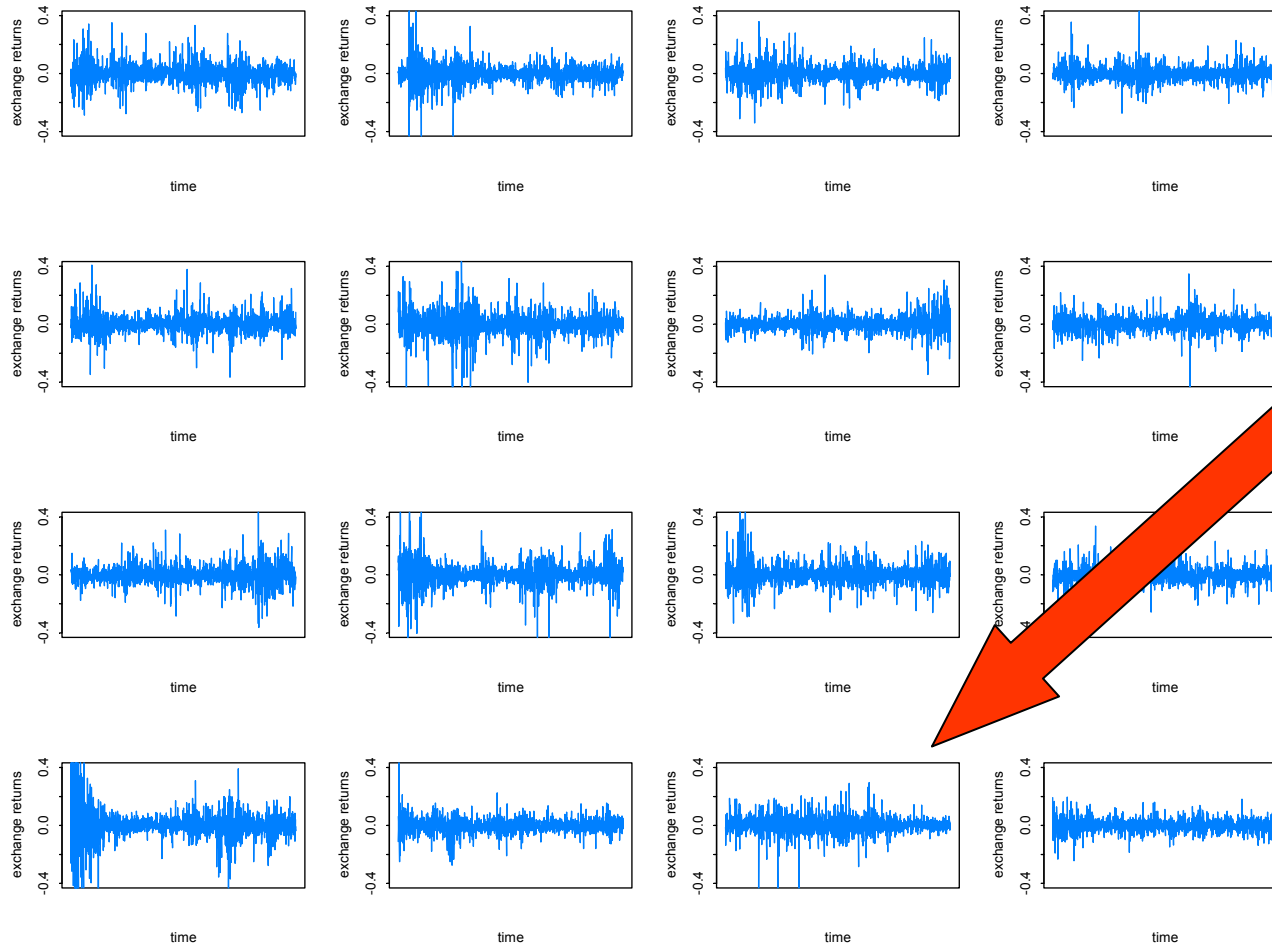


Example: Amazon-returns  
Hill's estimate of alpha (Hill Horror plots-Resnick)



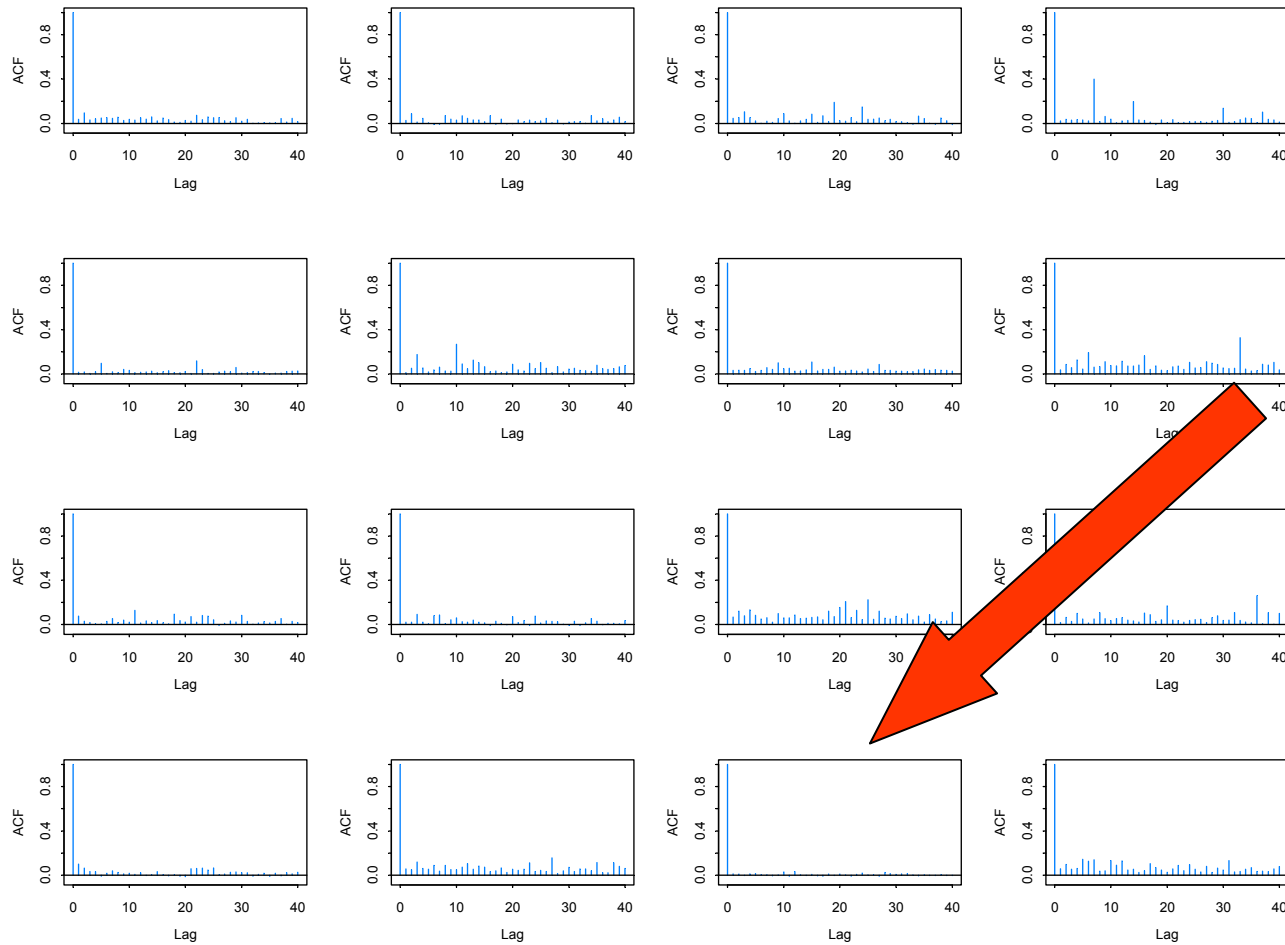
# Stărică Plots for the Amazon Data

15 realizations from GARCH model fitted to Amazon + exchange rate data. Which one is the real data?



## Stărică Plots for Amazon

ACF of the squares from the 15 realizations from the GARCH model on previous slide.



# Multiplicative models for log(returns)

## Basic model

$$\begin{aligned} X_t &= \ln(P_t) - \ln(P_{t-1}) \quad (\text{log returns}) \\ &= \sigma_t Z_t, \end{aligned}$$

where

- $\{Z_t\}$  is IID with mean 0, variance 1 (if exists). (e.g.  $N(0,1)$  or a  $t$ -distribution with  $\nu$  df.)
- $\{\sigma_t\}$  is the volatility process
- $\sigma_t$  and  $Z_t$  are independent.

## Properties:

- $EX_t = 0$ ,  $\text{Cov}(X_t, X_{t+h}) = 0$ ,  $h > 0$  (uncorrelated if  $\text{Var}(X_t) < \infty$ )
- conditional heteroscedastic (condition on  $\sigma_t$ ).



## Multiplicative models for log(returns)-cont

$$X_t = \sigma_t Z_t \quad (\text{observation eqn in state-space formulation})$$

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 .$$

Special case: ARCH(1):

$$\begin{aligned} X_t^2 &= (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 \\ &= \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \\ &= A_t X_{t-1}^2 + B_t \end{aligned} \quad (\text{stochastic recurrence eqn})$$

$$\rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3.$$

## Multiplicative models for log(returns)-cont

**GARCH(2,1):**  $X_t = \sigma_t Z_t$ ,  $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$  .

Then  $\mathbf{Y}_t = (\sigma_t^2, X_{t-1}^2)'$  follows the SRE given by

$$\begin{bmatrix} \sigma_t^2 \\ X_{t-1}^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_{t-1}^2 + \beta_1 & \alpha_2 \\ Z_{t-1}^2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{t-1}^2 \\ X_{t-2}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix}$$

### Questions:

- Existence of a unique stationary solution to the SRE?
- Regular variation of the joint distributions?

## Multiplicative models for log(returns)-cont

$$X_t = \sigma_t Z_t \text{ (observation eqn in state-space formulation)}$$

(ii) stochastic volatility process (parameter-driven specification)

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \quad \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2)$$

$$\rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4$$

Question:

- Joint distributions of process regularly varying if distr of  $Z_1$  is regularly varying?

## Two models for log(returns)-cont

GARCH(1,1):

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad \{Z_t\} \sim \text{IID}(0,1)$$

Stochastic Volatility:

$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2)$$

Main question:

What intrinsic features in the data (*if any*) can be used to discriminate between these two models?

## Regular variation — univariate case

Def: The random variable  $X$  is *regularly varying with index*  $\alpha$  if

$$P(|X| > tx) / P(|X| > t) \rightarrow x^{-\alpha} \text{ and } P(X > t) / P(|X| > t) \rightarrow p,$$

or, equivalently, if

$$P(X > tx) / P(|X| > t) \rightarrow px^{-\alpha} \text{ and } P(X < -tx) / P(|X| > t) \rightarrow qx^{-\alpha},$$

where  $0 \leq p \leq 1$  and  $p+q=1$ .

Equivalence:

$X$  is  $RV(\alpha)$  if and only if  $P(X \in t \bullet) / P(|X| > t) \rightarrow_v \mu(\bullet)$

( $\rightarrow_v$  vague convergence of measures on  $\mathbb{R} \setminus \{0\}$ ). In this case,

$$\mu(dx) = \left( p\alpha x^{-\alpha-1} I(x>0) + q\alpha (-x)^{-\alpha-1} I(x<0) \right) dx$$

**Note:**  $\mu(tA) = t^{-\alpha} \mu(A)$  for every  $t$  and  $A$  bounded away from 0.

## Regular variation — univariate case (cont)

Another formulation (polar coordinates):

Define the  $\pm 1$  valued rv  $\theta$ ,  $P(\theta = 1) = p$ ,  $P(\theta = -1) = 1 - p = q$ .

Then

$X$  is  $RV(\alpha)$  if and only if

$$\frac{P(|X| > t x, X/|X| \in S)}{P(|X| > t)} \rightarrow x^{-\alpha} P(\theta \in S)$$

or

$$\frac{P(|X| > t x, X/|X| \in \bullet)}{P(|X| > t)} \rightarrow_v x^{-\alpha} P(\theta \in \bullet)$$

( $\rightarrow_v$  vague convergence of measures on  $S^0 = \{-1, 1\}$ ).

## Regular variation — multivariate case

Multivariate regular variation of  $\mathbf{X}=(X_1, \dots, X_m)$ : There exists a random vector  $\theta \in S^{m-1}$  such that

$$P(|\mathbf{X}| > t x, \mathbf{X}/|\mathbf{X}| \in \bullet) / P(|\mathbf{X}| > t) \rightarrow_v x^{-\alpha} P(\theta \in \bullet)$$

( $\rightarrow_v$  vague convergence on  $S^{m-1}$ , unit sphere in  $\mathbb{R}^m$ ).

- $P(\theta \in \bullet)$  is called the **spectral measure**
- $\alpha$  is the **index of  $\mathbf{X}$** .

Equivalence:

$$\frac{P(\mathbf{X} \in t\bullet)}{P(|\mathbf{X}| > t)} \rightarrow_v \mu(\bullet)$$

$\mu$  is a measure on  $\mathbb{R}^m$  which satisfies for  $x > 0$  and  $A$  bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$

## Regular variation — multivariate case (cont)

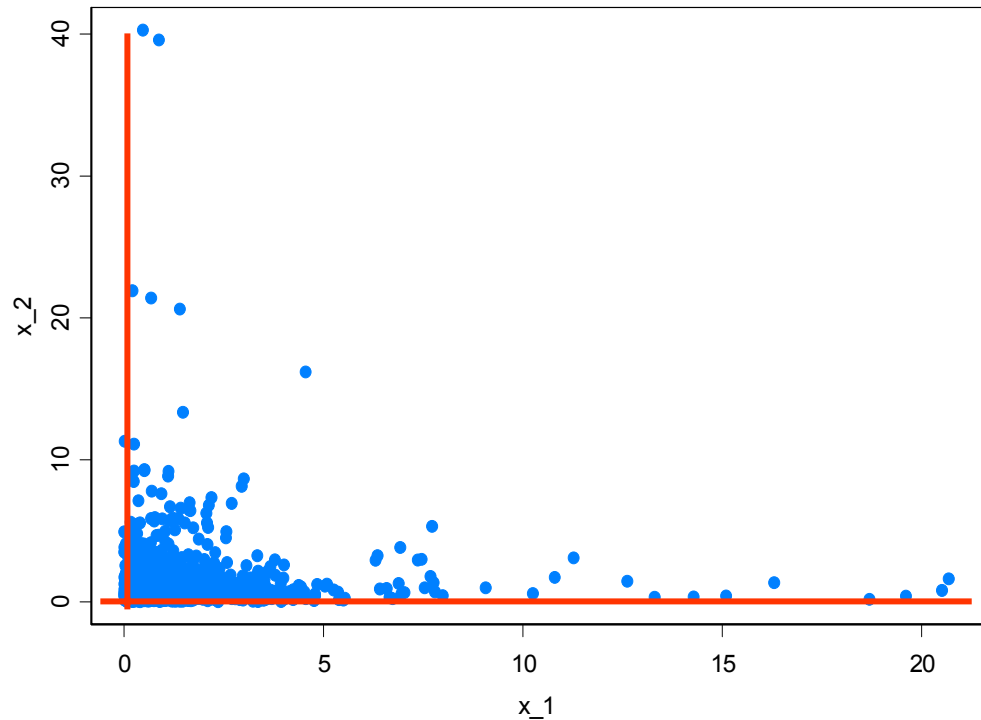
Examples:

1. If  $X_1 > 0$  and  $X_2 > 0$  are iid  $RV(\alpha)$ , then  $\mathbf{X} = (X_1, X_2)$  is multivariate regularly varying with index  $\alpha$  and *spectral distribution*

$$P(\theta = (0, 1)) = P(\theta = (1, 0)) = .5 \quad (\text{mass on axes}).$$

**Interpretation:** Unlikely that  $X_1$  and  $X_2$  are very large at the same time.

**Figure:** plot of  $(X_{t1}, X_{t2})$  for realization of 10,000.





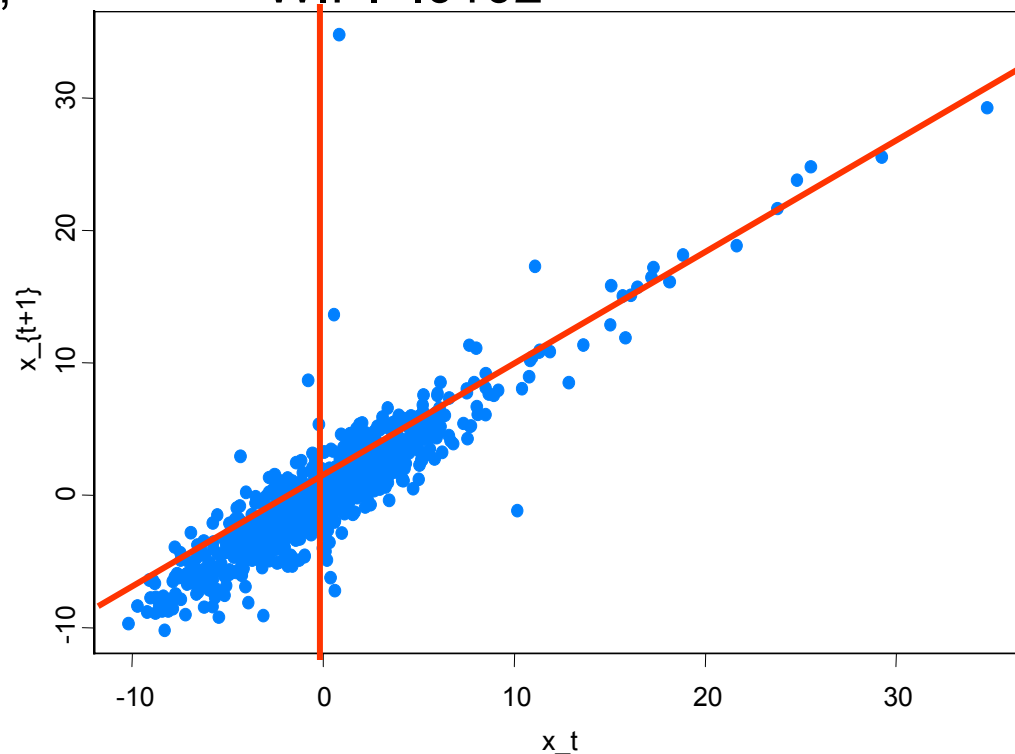
2. If  $X_1 = X_2 > 0$ , then  $\mathbf{X} = (X_1, X_2)$  is multivariate regularly varying with index  $\alpha$  and *spectral distribution*

$$P(\theta = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$

3. AR(1):  $X_t = .9 X_{t-1} + Z_t$ ,  $\{Z_t\} \sim \text{IID symmetric stable (1.8)}$

Distr of  $\theta$ :  $\begin{cases} \pm(1,.9)/\text{sqrt}(1.81), \text{ W.P. } .9898 \\ \pm(0,1), \\ \text{ W.P. } .0102 \end{cases}$

**Figure:** plot of  $(X_t, X_{t+1})$  for realization of 10,000.



## Applications of multivariate regular variation

- Domain of attraction for *sums of iid random vectors* (Rvaceva, 1962). That is, when does the partial sum

$$a_n^{-1} \sum_{t=1}^n \mathbf{X}_t$$

converge for some constants  $a_n$ ?

- *Spectral measure* of multivariate stable vectors.
- *Domain of attraction* for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

$$a_n^{-1} \bigvee_{t=1}^n \mathbf{X}_t$$

- Weak convergence of *point processes* with iid points.
- Solution to *stochastic recurrence equations*,  $\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t$
- Weak convergence of *sample autocovariances*.

## Applications of multivariate regular variation (cont)

### Linear combinations:

$\mathbf{X} \sim \text{RV}(\alpha) \Rightarrow$  all linear combinations of  $\mathbf{X}$  are regularly varying

i.e., there exist  $\alpha$  and slowly varying fcn  $L(\cdot)$ , s.t.

$$P(\mathbf{c}^T \mathbf{X} > t) / (t^\alpha L(t)) \rightarrow w(\mathbf{c}), \text{ exists for all real-valued } \mathbf{c},$$

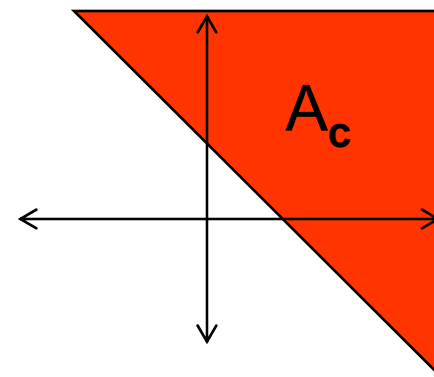
where

$$w(t\mathbf{c}) = t^{-\alpha} w(\mathbf{c}).$$

Use vague convergence with  $A_{\mathbf{c}} = \{\mathbf{y} : \mathbf{c}^T \mathbf{y} > 1\}$ , i.e.,

$$\frac{P(\mathbf{X} \in tA_{\mathbf{c}})}{t^{-\alpha} L(t)} = \frac{P(\mathbf{c}^T \mathbf{X} > t)}{P(|\mathbf{X}| > t)} \rightarrow \mu(A_{\mathbf{c}}) =: w(\mathbf{c}),$$

where  $t^\alpha L(t) = P(|\mathbf{X}| > t)$ .



## Applications of multivariate regular variation (cont)

### Converse?

$\mathbf{X} \sim \text{RV}(\alpha) \iff$  all linear combinations of  $\mathbf{X}$  are regularly varying?

There exist  $\alpha$  and slowly varying fcn  $L(\cdot)$ , s.t.

(LC)  $P(\mathbf{c}^T \mathbf{X} > t) / (t^\alpha L(t)) \rightarrow w(\mathbf{c})$ , exists for all real-valued  $\mathbf{c}$ .

Theorem (Basrak, Davis, Mikosch, '02). Let  $\mathbf{X}$  be a random vector.

1. If  $\mathbf{X}$  satisfies (LC) with  $\alpha$  non-integer, then  $\mathbf{X}$  is  $\text{RV}(\alpha)$ .
2. If  $\mathbf{X} > 0$  satisfies (LC) for non-negative  $\mathbf{c}$  and  $\alpha$  is non-integer, then  $\mathbf{X}$  is  $\text{RV}(\alpha)$ .
3. If  $\mathbf{X} > 0$  satisfies (LC) with  $\alpha$  an odd integer, then  $\mathbf{X}$  is  $\text{RV}(\alpha)$ .

## Applications of multivariate regular variation (cont)

There exist  $\alpha$  and slowly varying fcn  $L(\cdot)$ , s.t.

(LC)  $P(\mathbf{c}^T \mathbf{X} > t) / (t^\alpha L(t)) \rightarrow w(\mathbf{c})$ , exists for all real-valued  $\mathbf{c}$ .

1. If  $\mathbf{X}$  satisfies (LC) with  $\alpha$  non-integer, then  $\mathbf{X}$  is  $RV(\alpha)$ .
2. If  $\mathbf{X} > 0$  satisfies (LC) for non-negative  $\mathbf{c}$  and  $\alpha$  is non-integer, then  $\mathbf{X}$  is  $RV(\alpha)$ .
3. If  $\mathbf{X} > 0$  satisfies (LC) with  $\alpha$  an odd integer, then  $\mathbf{X}$  is  $RV(\alpha)$ .

Remark: Hult and Lindskog (2005) show that .

- 1 cannot be extended to integer  $\alpha$ .
- 2 cannot be extended to integer  $\alpha$ .
- It is unknown if 3 can be extended to even integers.

## Applications of theorem

1. **Kesten (1973)**. Under general conditions, **(LC)** holds with  $L(t)=1$  for stochastic recurrence equations of the form

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID},$$

$\mathbf{A}_t$   $d \times d$  random matrices,  $\mathbf{B}_t$  random  $d$ -vectors.

It follows that the distributions of  $\mathbf{Y}_t$ , and in fact all of the finite dim'l distrs of  $\mathbf{Y}_t$  are regularly varying (if  $\alpha$  is non-even).

2. **GARCH processes**. Since squares of a GARCH process can be embedded in a SRE, the *finite dimensional distributions* of a **GARCH** are regularly varying.

## Examples

Example of ARCH(1):  $X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t$ ,  $\{Z_t\} \sim \text{IID}$ .

$\alpha$  found by solving  $E|\alpha_1 Z^2|^{\alpha/2} = 1$ .

$\alpha_1$	.312	.577	1.00	1.57
$\alpha$	8.00	4.00	2.00	1.00

Distr of  $\theta$ :

$$P(\theta \in \bullet) = E\{ \|(B,Z)\|^\alpha I(\arg((B,Z)) \in \bullet) \} / E\|(B,Z)\|^\alpha$$

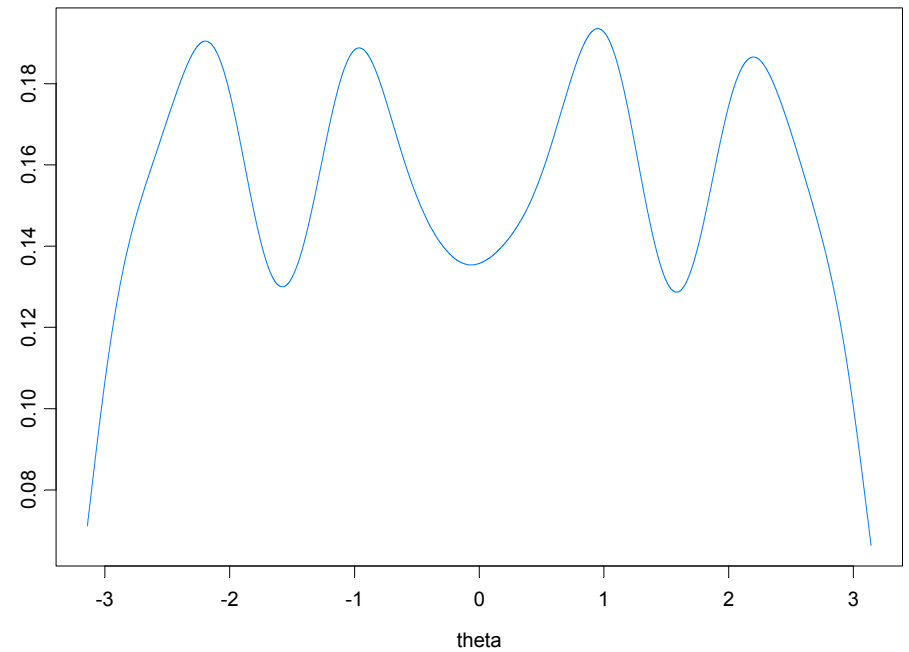
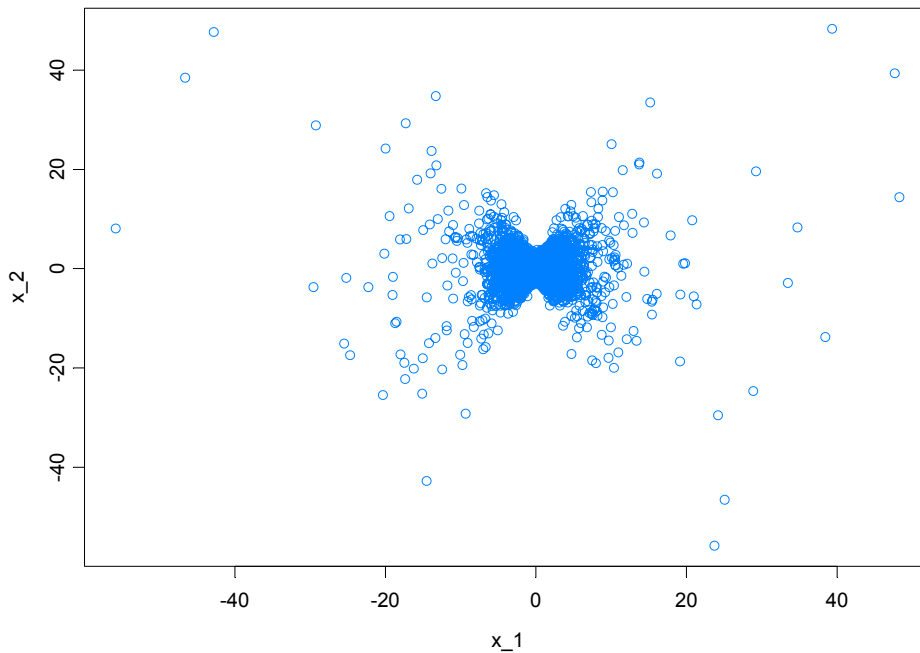
where

$$P(B = 1) = P(B = -1) = .5$$

## Examples (cont)

Example of ARCH(1):  $\alpha_0=1, \alpha_1=1, \alpha=2, X_t=(\alpha_0+\alpha_1 X_{t-1}^2)^{1/2}Z_t, \{Z_t\}\sim\text{IID}$

Figures: plots of  $(X_t, X_{t+1})$  and estimated distribution of  $\theta$  for realization of 10,000.





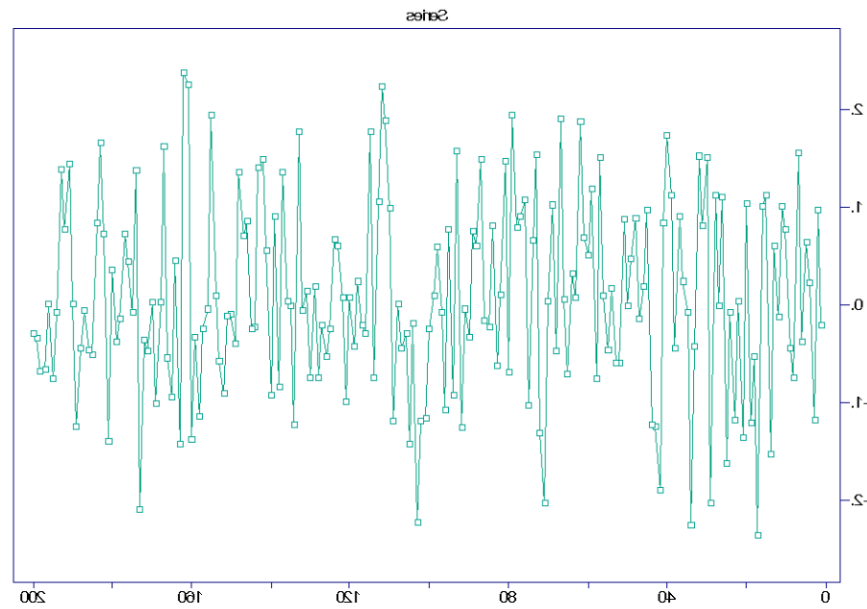
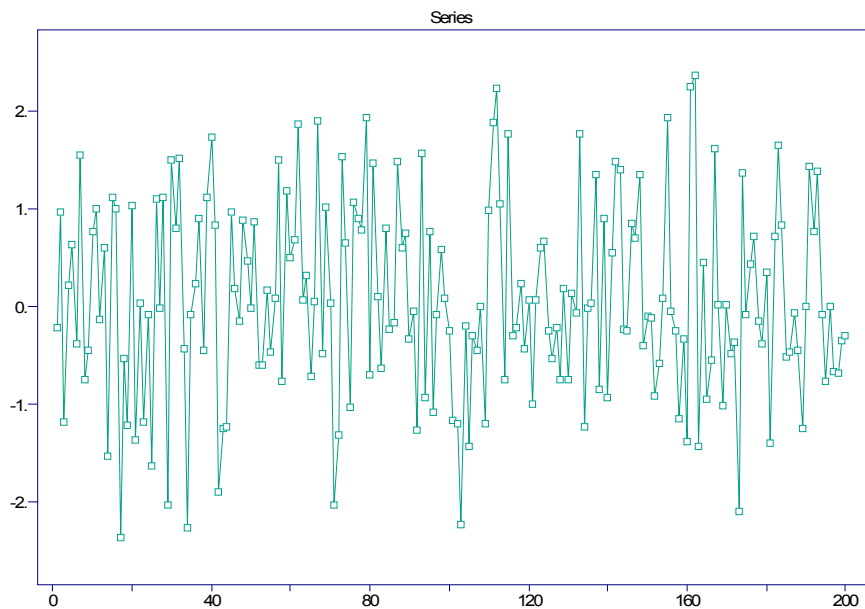
## Excursion to time-reversibility

**Reversibility.** A stationary sequence of random variables  $\{X_t\}$  is *time-reversible* if  $(X_1, \dots, X_n) =_d (X_n, \dots, X_1)$  for all  $n > 1$ .

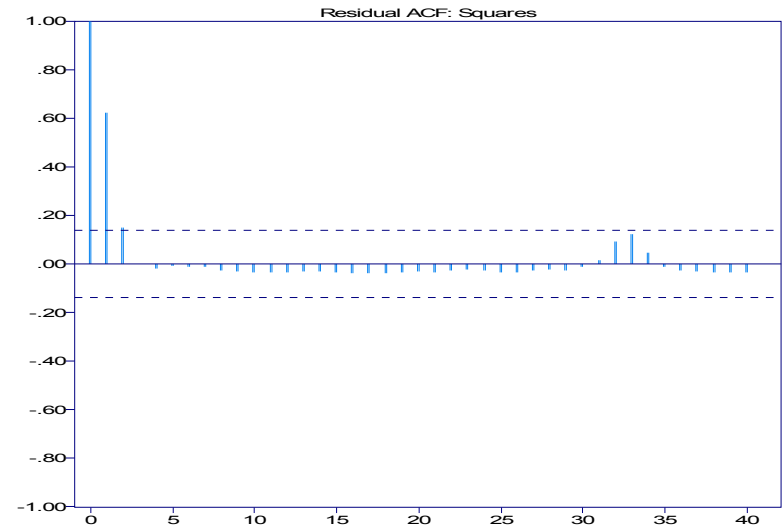
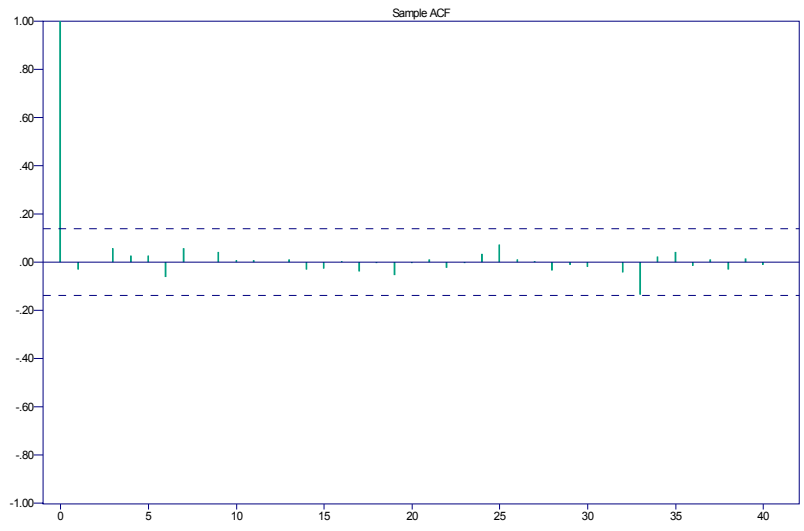
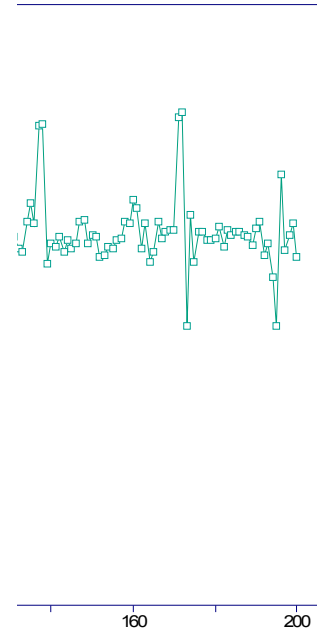
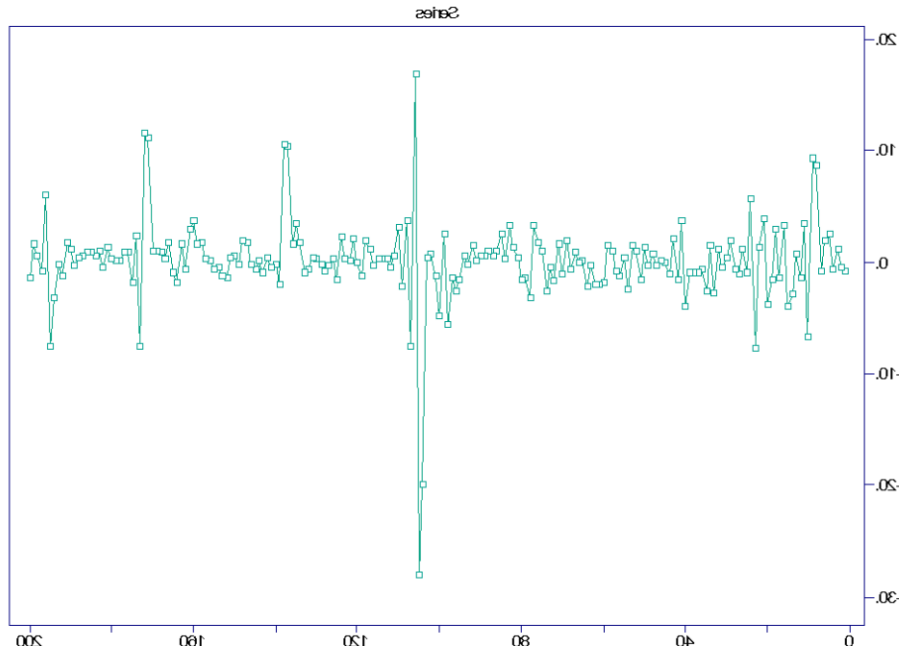
**Results:** i) IID sequences  $\{Z_t\}$  are time-reversible.

ii) Linear time series (with a couple obvious exceptions) are time-reversible iff *Gaussian*. (Breidt and Davis '91)

**Application:** If plot of time series does not look time-reversible, then it *cannot* be modeled as IID or a Gaussian process. Use the “flip and compare” inspection test!



# Reversibility. Does the following series look time-reversible?

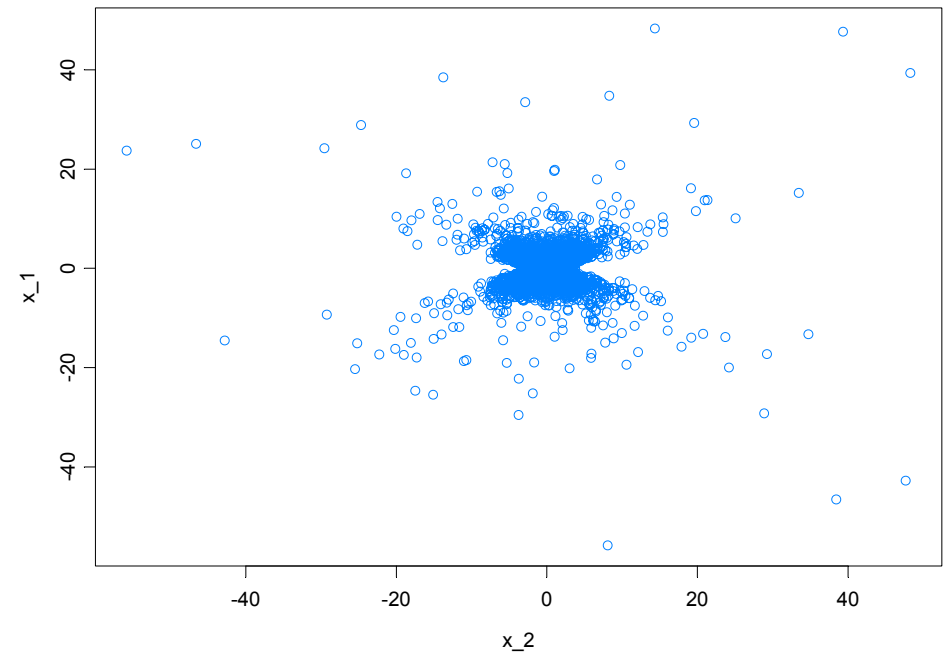
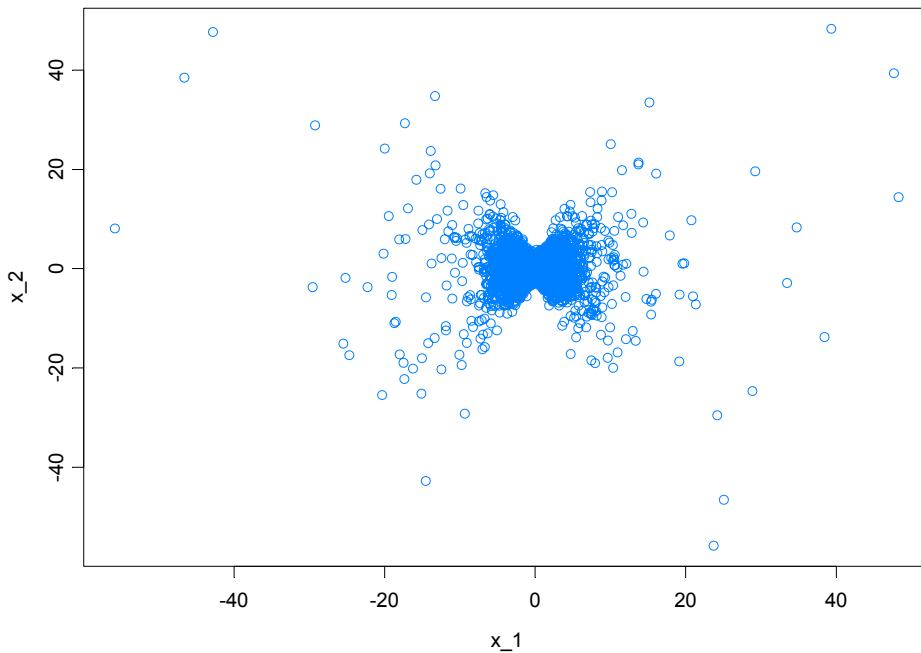


## Examples (cont)

Example of ARCH(1):  $\alpha_0=1, \alpha_1=1, \alpha=2, X_t=(\alpha_0+\alpha_1 X_{t-1}^2)^{1/2}Z_t, \{Z_t\}\sim\text{IID}$

Is this process time-reversible?

Figures: plots of  $(X_t, X_{t+1})$  and  $(X_{t+1}, X_t)$  implies *non-reversible*.



## Examples (cont)

Example: SV model  $X_t = \sigma_t Z_t$

Suppose  $Z_t \sim \text{RV}(\alpha)$  and

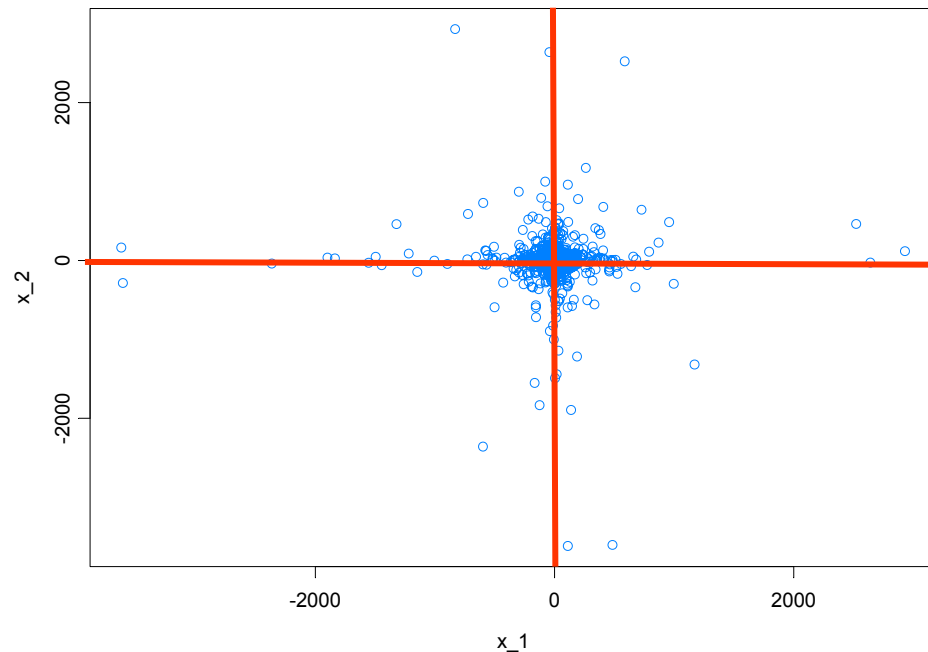
$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2)$$

Then  $\mathbf{Z}_n = (Z_1, \dots, Z_n)'$  is regular varying with index  $\alpha$  and so is

$$\mathbf{X}_n = (X_1, \dots, X_n)' = \text{diag}(\sigma_1, \dots, \sigma_n) \mathbf{Z}_n$$

with spectral distribution concentrated on  $(\pm 1, 0)$ ,  $(0, \pm 1)$ .

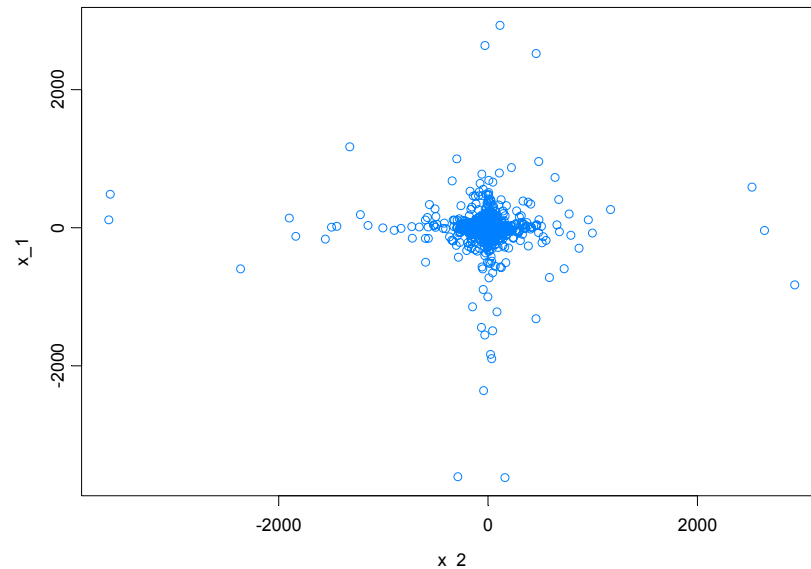
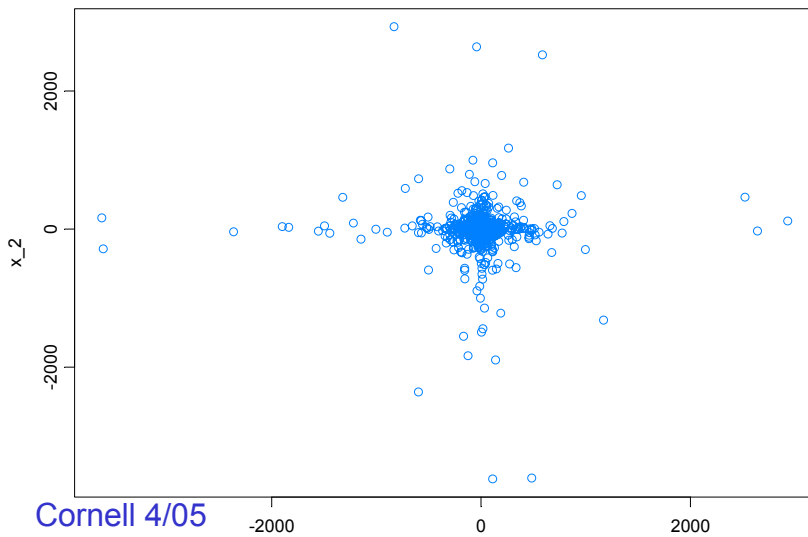
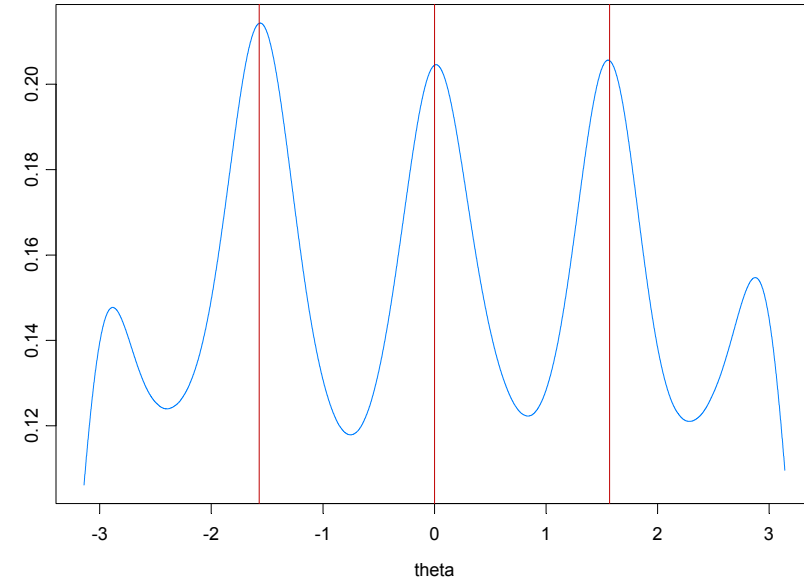
Figure: plot of  
 $(X_t, X_{t+1})$  for  
realization of 10,000.



## Examples (cont)

Example: SV model  $X_t = \sigma_t Z_t$

- SV processes are time-reversible if log-volatility is Gaussian.
- Asymptotically time-reversible if log-volatility is nonGaussian



## Point process convergence

**Theorem** (Davis & Hsing '95, Davis & Mikosch '97). Let  $\{\mathbf{X}_t\}$  be a stationary sequence of random  $m$ -vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let  $(\theta_{-k}, \dots, \theta_k)$  be the vector in  $\mathcal{S}^{(2k+1)m-1}$  in the definition).

(ii) mixing condition  $A(a_n)$  or strong mixing.

(iii)  $\limsup_{k \rightarrow \infty} \liminf_{n \rightarrow \infty} P(\bigvee_{k \leq |t| \leq r_n} |\mathbf{X}_t| > a_n y \mid |\mathbf{X}_0| > a_n y) = 0$ .

Then

$$\gamma = \lim_{k \rightarrow \infty} E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|)_+ / E|\theta_0^{(k)}|^\alpha \quad (\text{extremal index})$$

exists. If  $\gamma > 0$ , then

$$N_n := \sum_{t=1}^n \varepsilon_{\mathbf{X}_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i Q_{ij}},$$

## Point process convergence(cont)

- $(P_i)$  are points of a Poisson process on  $(0, \infty)$  with intensity function

$$v(dy) = \gamma \alpha y^{-\alpha-1} dy.$$

- $\sum_{j=1}^{\infty} \varepsilon_{Q_{ij}}$ ,  $i \geq 1$ , are iid point process with distribution  $Q$ , and  $Q$  is the weak limit of

$$\lim_{k \rightarrow \infty} E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|)_+ I.(\sum_{|t| \leq k} \varepsilon_{\theta_t^{(k)}}) / E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|)_+$$

### Remarks:

1. GARCH and SV processes satisfy the conditions of the theorem.
2. Limit distribution for sample extremes and sample ACF follows from this theorem.

## Extremes for GARCH and SV processes

### Setup

- $X_t = \sigma_t Z_t$ ,  $\{Z_t\} \sim \text{IID}(0,1)$
- $X_t$  is RV ( $\alpha$ )
- Choose  $\{b_n\}$  s.t.  $nP(X_t > b_n) \rightarrow 1$

Then

$$P^n(b_n^{-1} X_1 \leq x) \rightarrow \exp\{-x^{-\alpha}\}.$$

Then, with  $M_n = \max\{X_1, \dots, X_n\}$ ,

(i) GARCH:

$$P(b_n^{-1} M_n \leq x) \rightarrow \exp\{-\gamma x^{-\alpha}\},$$

$\gamma$  is extremal index ( $0 < \gamma < 1$ ).

(ii) SV model:

$$P(b_n^{-1} M_n \leq x) \rightarrow \exp\{-x^{-\alpha}\},$$

extremal index  $\gamma = 1$  no clustering.



## Extremes for GARCH and SV processes (cont)

(i) GARCH:  $P(b_n^{-1}M_n \leq x) \rightarrow \exp\{-\gamma x^{-\alpha}\}$

(ii) SV model:  $P(b_n^{-1}M_n \leq x) \rightarrow \exp\{-x^{-\alpha}\}$

### Remarks about extremal index.

(i)  $\gamma < 1$  implies clustering of exceedances

(ii) Numerical example. Suppose  $c$  is a threshold such that

$$P^n(b_n^{-1}X_1 \leq c) \sim .95$$

Then, if  $\gamma = .5$ ,  $P(b_n^{-1}M_n \leq c) \sim (.95)^{.5} = .975$

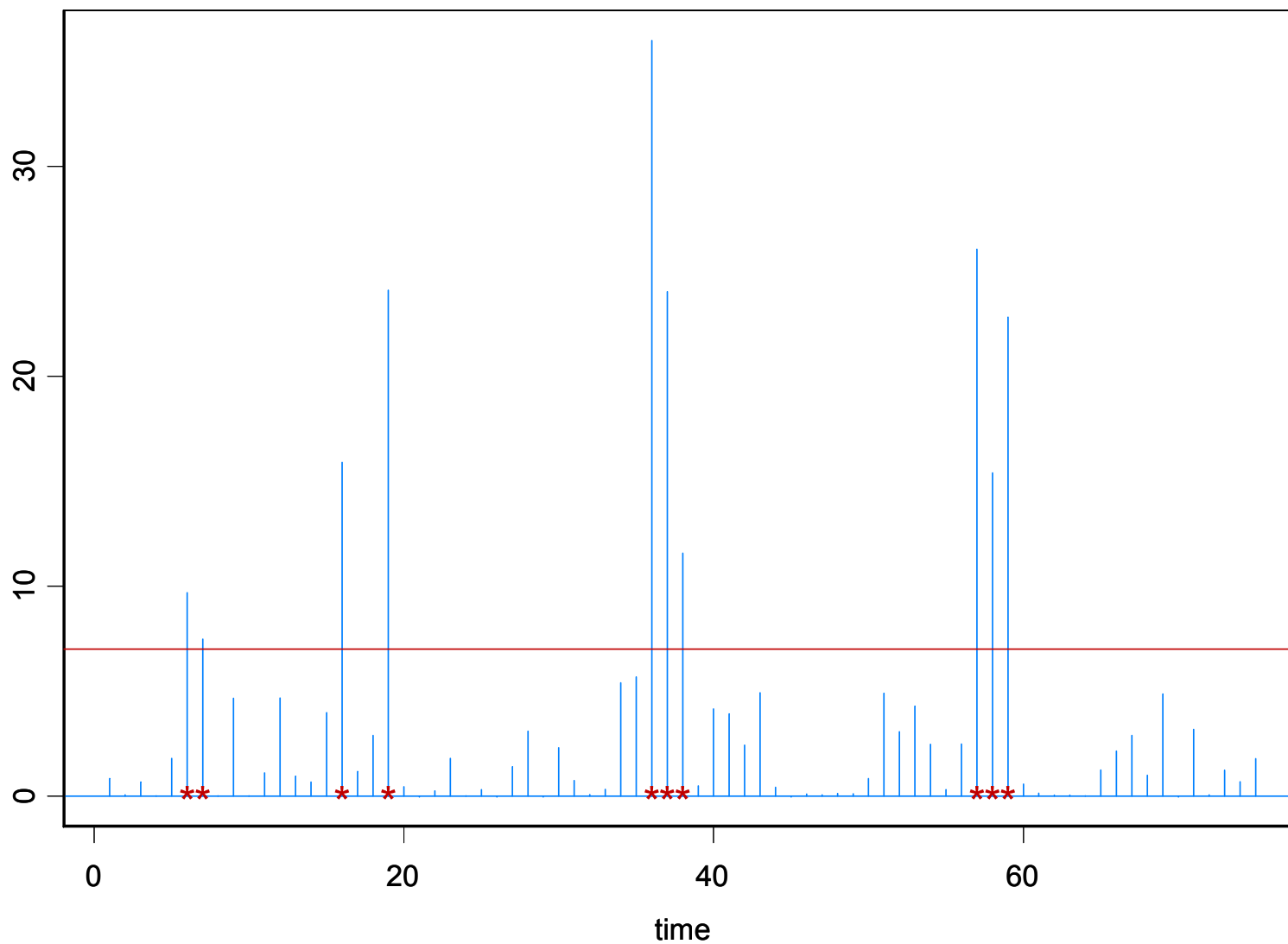
(iii)  $1/\gamma$  is the *mean cluster size* of exceedances.

(iv) Use  $\gamma$  to *discriminate* between GARCH and SV models.

(v) Even for the light-tailed SV model (i.e.,  $\{Z_t\} \sim \text{IID } N(0,1)$ ), the *extremal index* is 1 (see Breidt and Davis '98 )

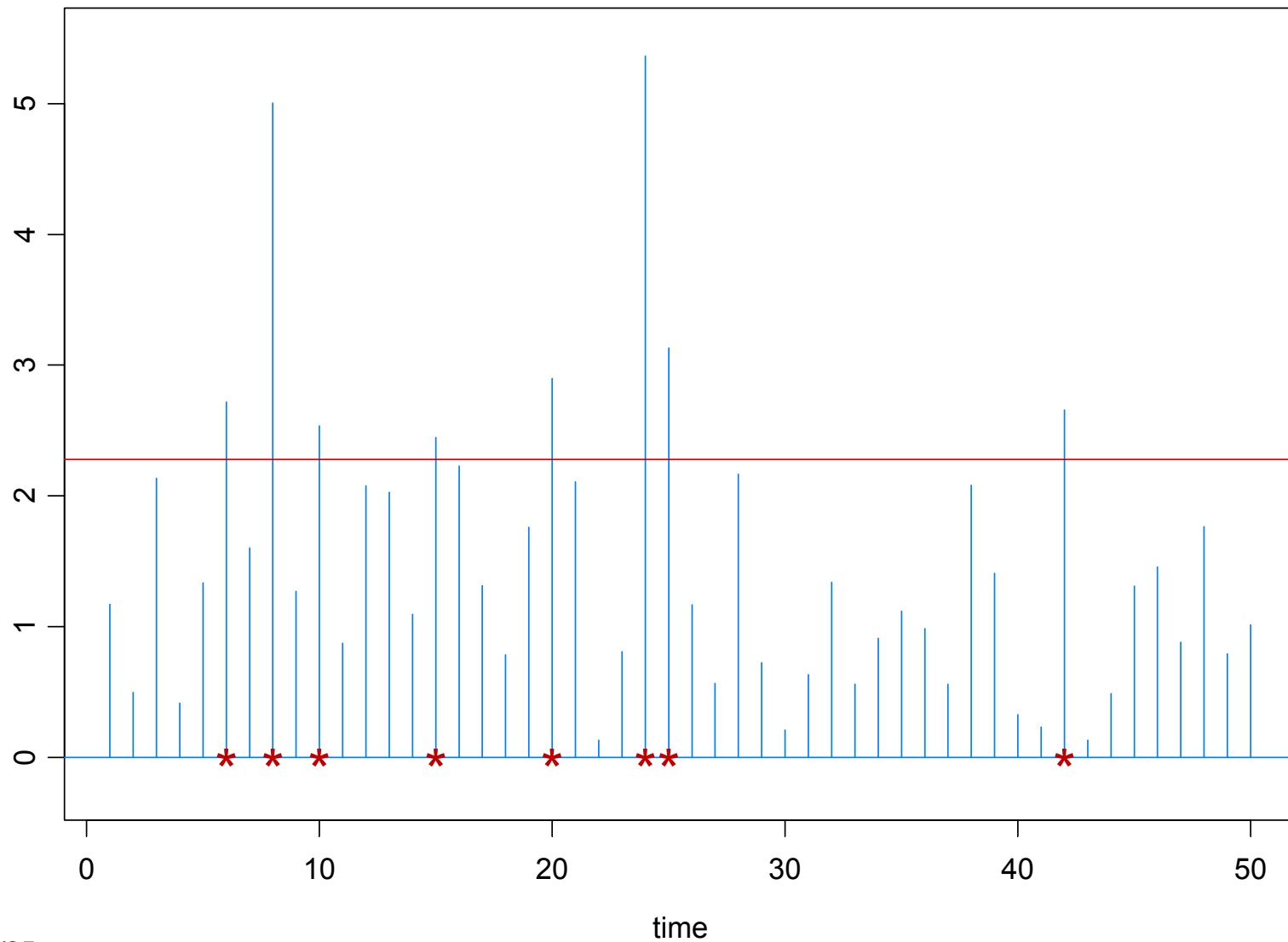
# Extremes for GARCH and SV processes (cont)

## Absolute values of ARCH

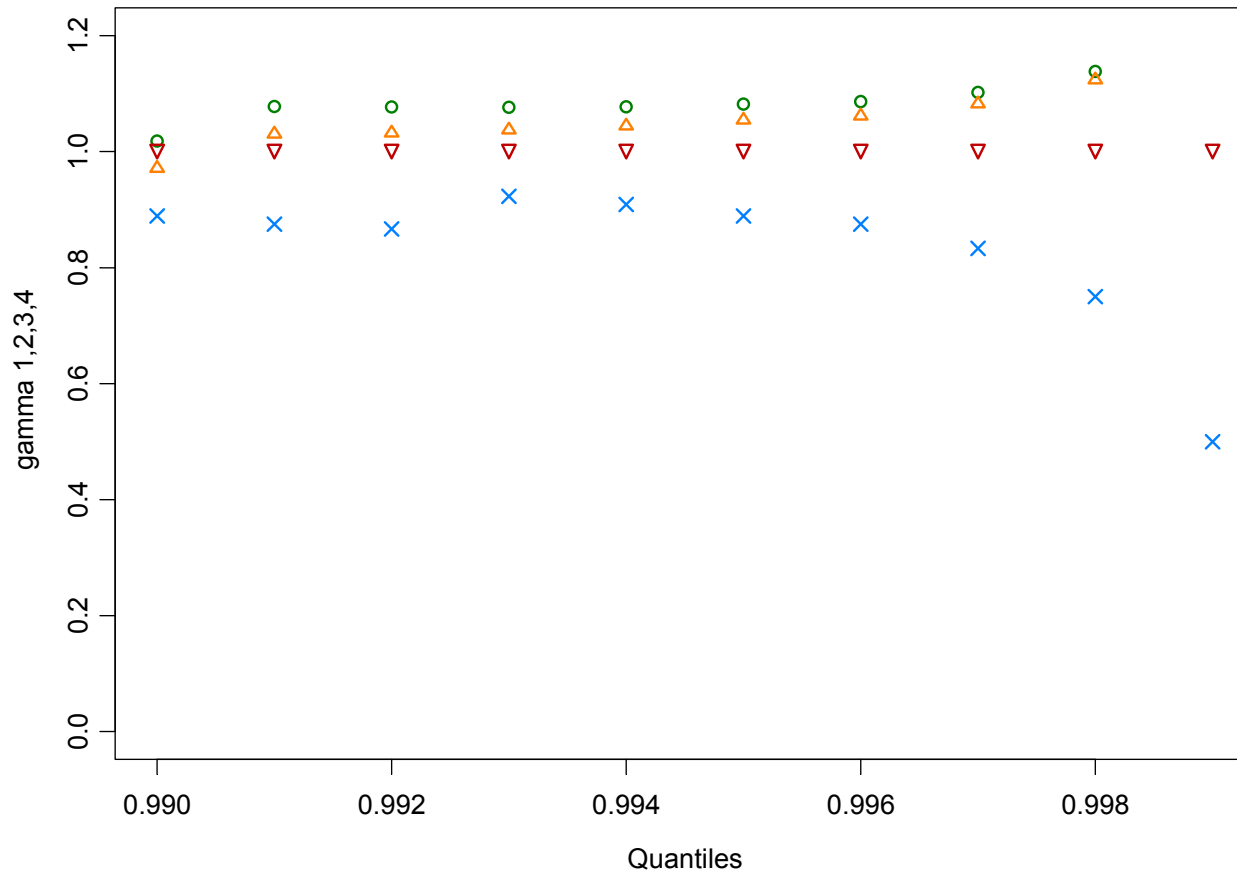


# Extremes for GARCH and SV processes (cont)

Absolute values of SV process



# Extremal Index Estimates for Amazon



$\gamma_1$  = block method

$\gamma_2$  = 1/(mean cluster size)

$\gamma_3$  = interval method (Ferro and Segers)

$\gamma_4$  = interval method (Ferro and Segers)

## Summary of results for ACF of GARCH(p,q) and SV models

### GARCH(p,q)

$\alpha \in (0,2)$ :

$$(\hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\dots,m},$$

$\alpha \in (2,4)$ :

$$(n^{1-2/\alpha} \hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)(V_h)_{h=1,\dots,m}.$$

$\alpha \in (4,\infty)$ :

$$(n^{1/2} \hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)(G_h)_{h=1,\dots,m}.$$

**Remark:** Similar results hold for the sample ACF based on  $|X_t|$  and  $X_t^2$ .

## Summary of results for ACF of GARCH(p,q) and SV models (cont)

### SV Model

$\alpha \in (0, 2)$ :

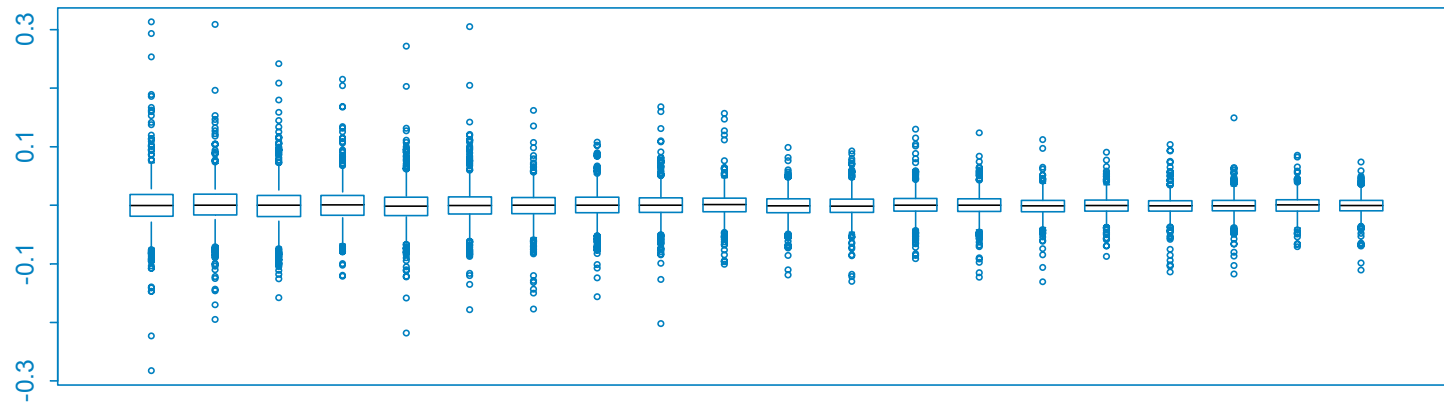
$$(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

$\alpha \in (2, \infty)$ :

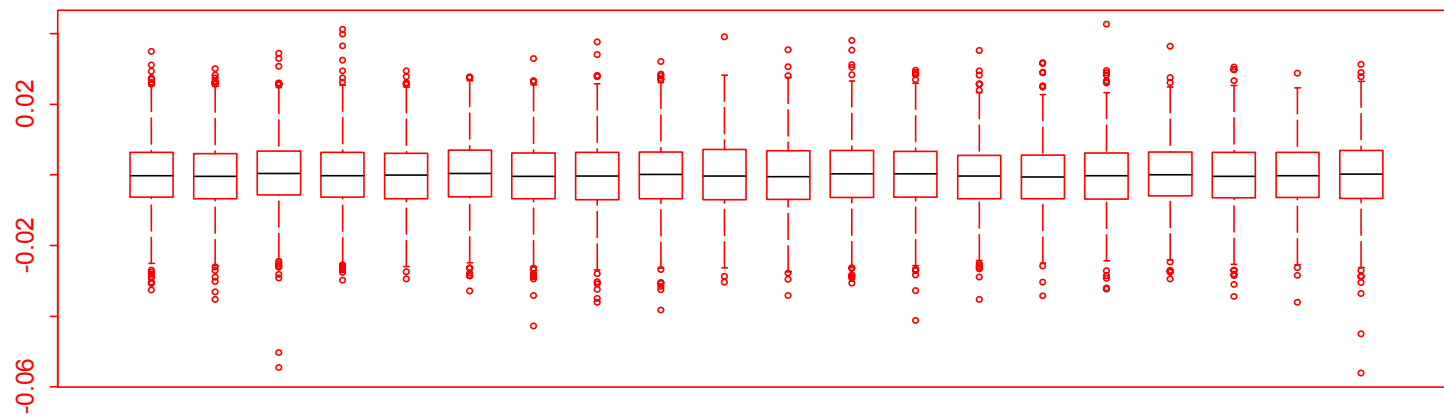
$$\left( n^{1/2} \hat{\rho}_X(h) \right)_{h=1, \dots, m} \xrightarrow{d} \gamma_X^{-1}(0) (G_h)_{h=1, \dots, m}.$$

# Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

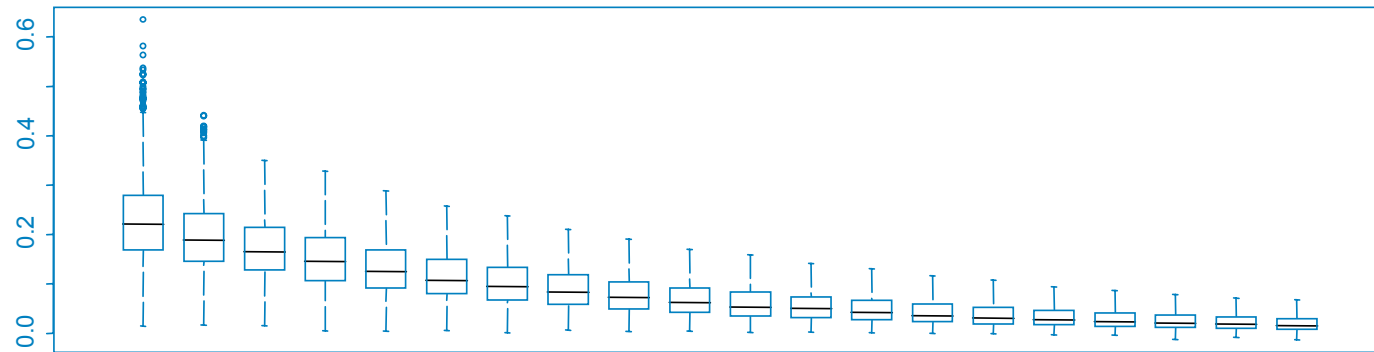


(b) SV Model, n=10000

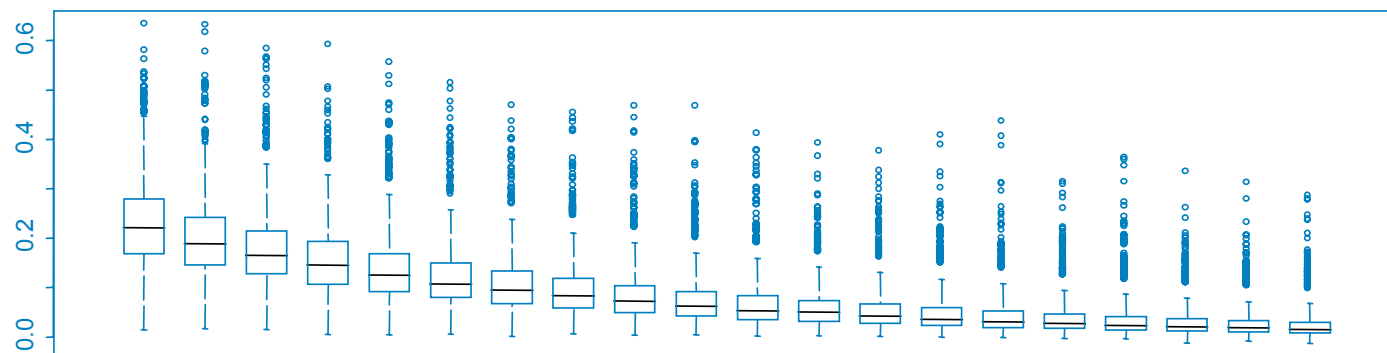


# Sample ACF for Squares of GARCH (1000 reps)

(a) GARCH(1,1) Model, n=10000



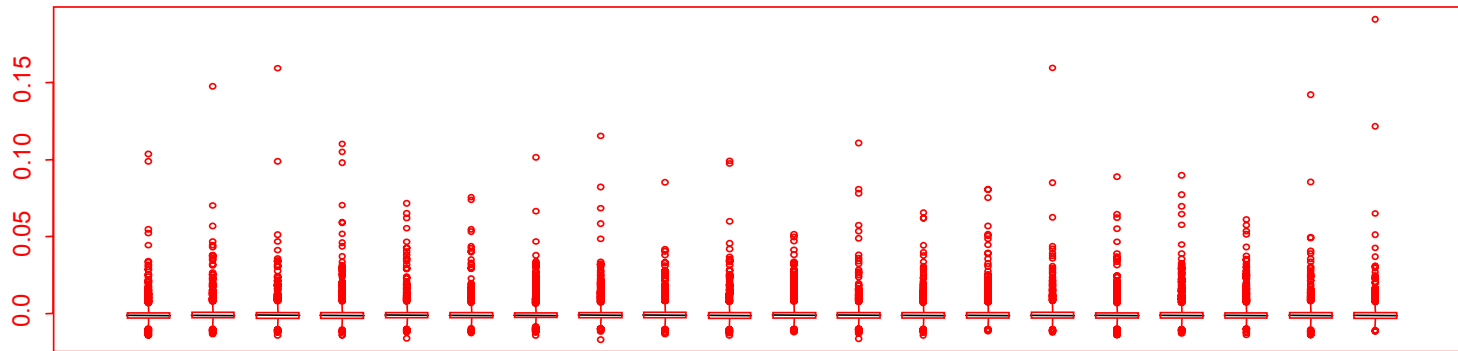
b) GARCH(1,1) Model, n=100000



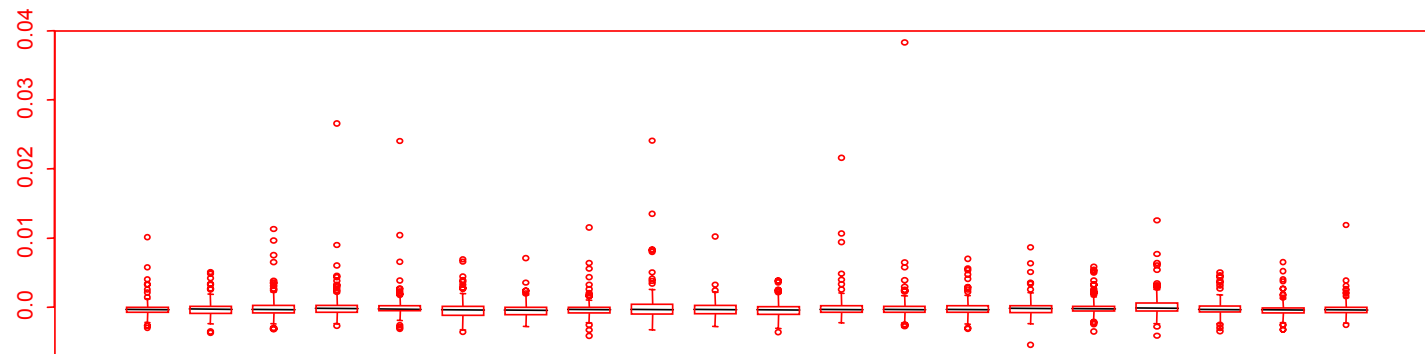


# Sample ACF for Squares of SV (1000 reps)

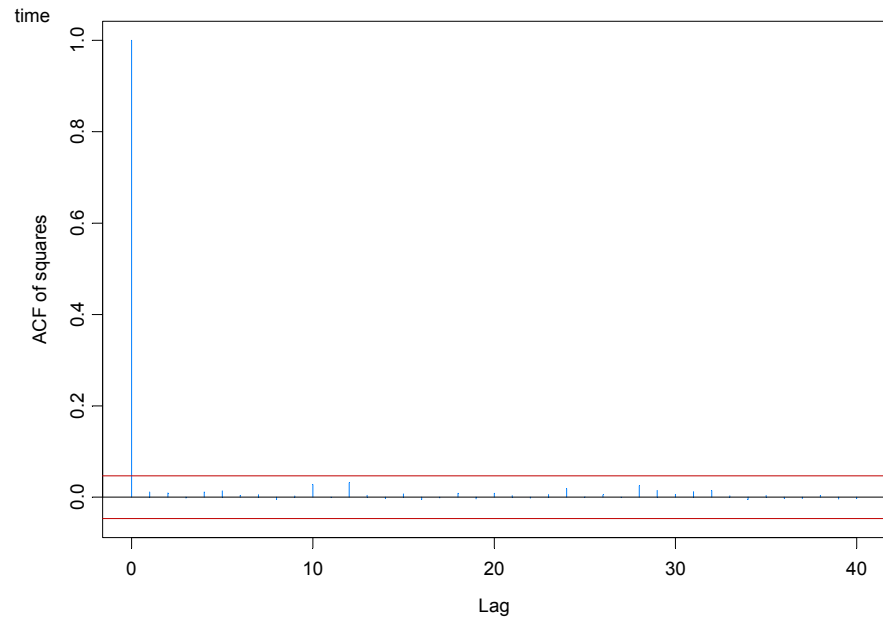
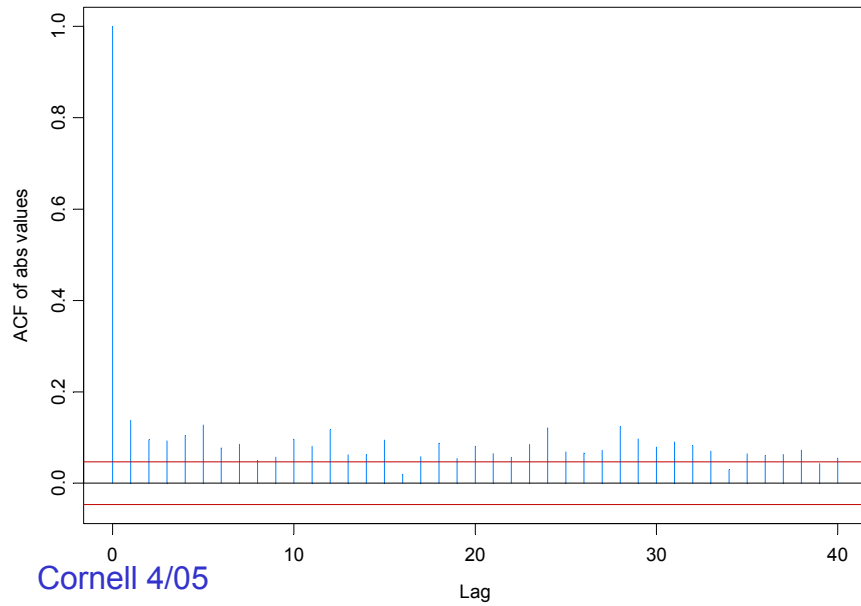
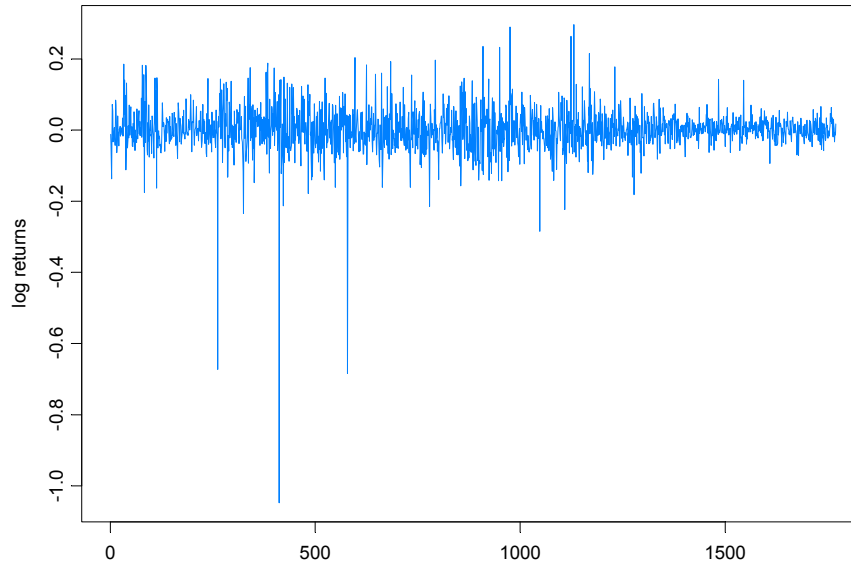
(c) SV Model, n=10000



(d) SV Model, n=100000



# Example: Amazon-returns (May 16, 1997 – June 16, 2004)

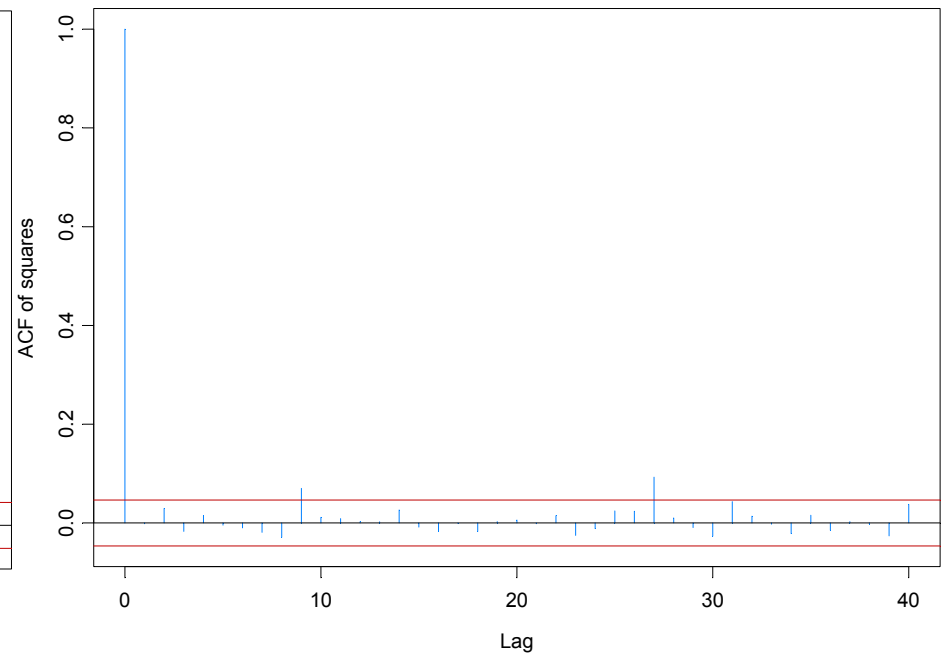
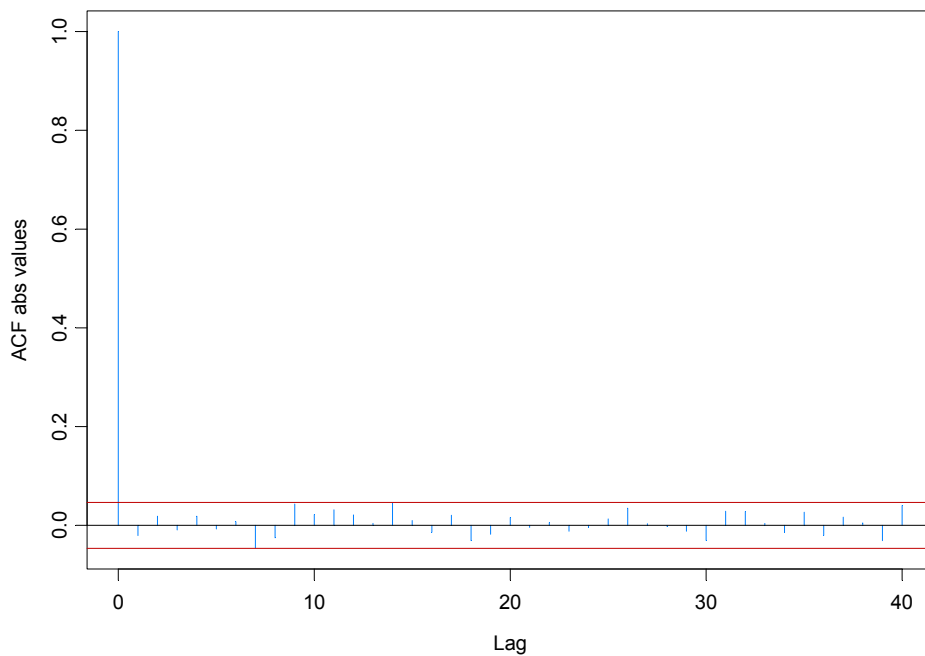


## Amazon returns (GARCH model)

GARCH(1,1) model fit to Amazon returns:

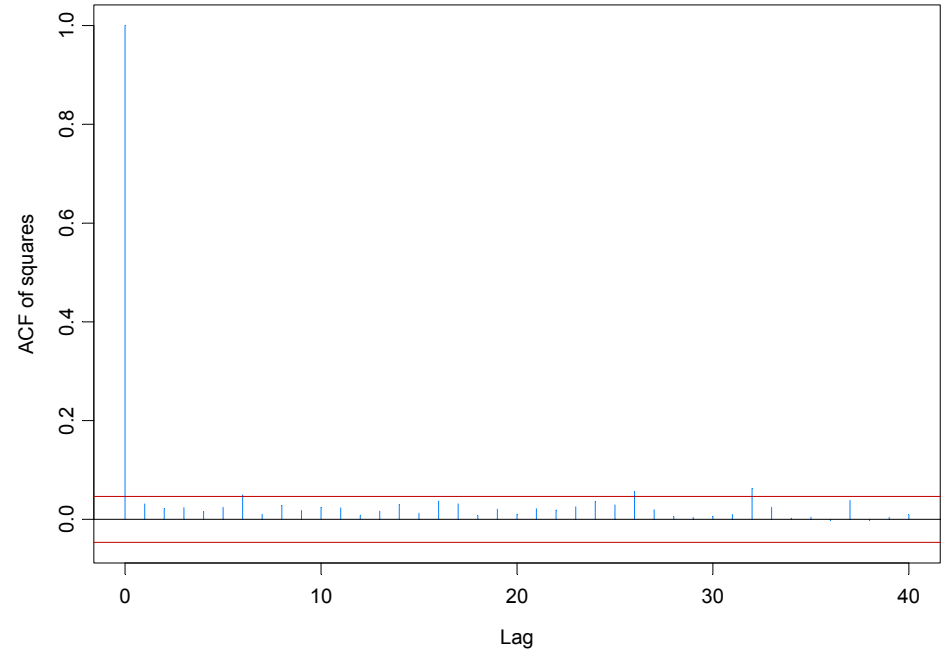
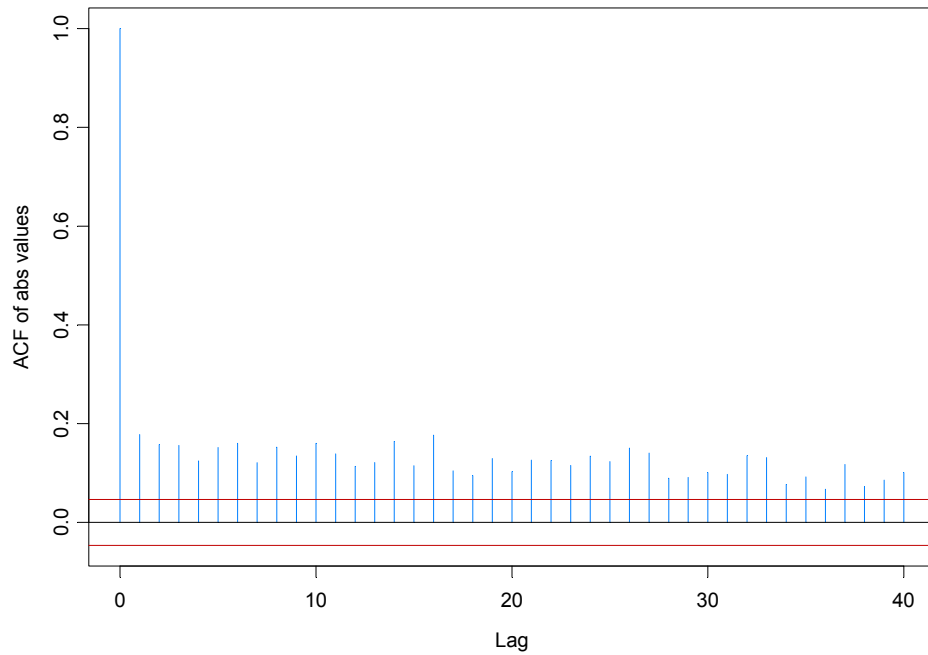
$$\alpha_0 = .00002493, \alpha_1 = .0385, \beta_1 = .957, X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \{Z_t\} \sim \text{IID } t(3.672)$$

Simulation from GARCH(1,1) model



# Amazon returns (SV model)

Stochastic volatility model fit to Amazon returns:



## Wrap-up

- *Regular variation* is a flexible tool for modeling both *dependence* and *tail heaviness*.
- Useful for establishing *point process convergence* of heavy-tailed time series.
- *Extremal index*  $\gamma < 1$  for GARCH and  $\gamma = 1$  for SV.
- ACF has faster convergence for SV.